Fixed-Income Securities
Lecture 6: Symphony of In-class Exercises
Xtreme Review

Philip H. Dybvig
Washington University in Saint Louis

- Concepts

- Definitions

- Formulas

- Practice practice practice

Some Central Ideas

- arbitrage: exact and approximate

- pricing using replication

- hedging using replication

- formulas and intuitions relating rates

- traditional approaches

  - matching cash flows
  - duration
  - effective duration (mean reversion, slope and twist of yield curve)

- modern option pricing tools

  - binomial model (fast, single source of noise)
  - simulation (slow, many sources of noise, not good for American)

Basic Notation

spot rate r_t: quoted at $t - 1$ for borrowing/lending from $t - 1$ to t

forward rate $f(s, t)$: quoted at s for borrowing/lending from $t - 1$ to t

discount factor $D(s, t)$: price at s of receiving 1 at a future date t

zero-coupon rate $z(s, t)$: yield at s for a zero-coupon bond maturing at t

par coupon rate $c(s, t)$: coupon rate ($=$ yield) quoted at s for a coupon bond maturing at t and trading at par

present value PV: value today of a series of future cash flows

present value NPV: value today of a series of future cash flows, less the initial price

Basic Formulas

$$r_t = f(t - 1, t)$$

$$D(s, t) = \frac{1}{\Pi_{s=1}^{t}(1 + f(0, s))}$$

$$f(s, t) = \frac{D(s, t - 1)}{D(s, t)} - 1$$

$$D(s, t) = \frac{1}{(1 + z(s, t))^{t-s}}$$

$$z(s, t) = D(s, t)^{-(t-s)} - 1$$

$$c(s, T) = \frac{1 - D(s, T)}{\Sigma_{t=s+1}^{T} D(s, t)}$$

$$PV = \sum_{s=1}^{t} D(0, s)c_s$$

$$NPV = PV - P$$
$$= \sum_{s=0}^{t} D(0, s)c_s$$

In-class Exercise: computations using zero-coupon bond prices

Suppose a zero-coupon bond maturing one period out (at time 1) has a price of $90 and a zero-coupon bond maturing two periods out (at time 2) has a price of $80, both per $100 of face value.

1. Compute the discount factors $D(0,1)$ and $D(0,2)$.

2. Compute the forward rates $f(0,1)$ and $f(0,2)$.

3. If we can borrow forward at 10% from one year out until two periods out, what is the arbitrage?

In-class Exercise: computations using zero-coupon rates

Suppose the yield on a one-year discount bond is 10% and the yield on a two-year zero-coupon bond is 12%.

1. Compute the discount factors $D(0,1)$ and $D(0,2)$.

2. Compute the forward rates $f(0,1)$ and $f(0,2)$.

3. What is the price of a two-year coupon bond with a face value of 500 and a coupon rate of 20%?

In-class Exercise: matching cash flows

Suppose the price is $212 for a 2-year coupon bond with face of $200 and an annual coupon (first one one year from now) of $40. Suppose also that the price is $150 for a 1-year coupon bond with face of $150 and an annual coupon (one remaining one year from now) of $15.

Remaining pension benefits in a plan having two more years to go are $95,000 one year from now and $60,000 two years from now. What replicating portfolio of the two coupon bonds covers the pension liabilities exactly? What is the price of the replicating portfolio?

Formulas connecting rates

$$z(0, T) = \left(\prod_{s=1}^{T} (1 + f(0, s)) \right)^{1/T} - 1 \approx \frac{1}{T} \sum_{s=1}^{T} f(0, s)$$

The zero-coupon rate is an average of forward rates up to that maturity.

$$c(0, T) = \sum_{s=1}^{T} w(0, s) f(0, s)$$

where

$$w(0, s) = \frac{D(0, s)}{\Sigma_{t=1}^{T} D(0, t)}$$

The par-coupon rate is a weighted average of forward rates up to that date, with more weight on the earlier maturities.

In-class Exercise: formulas connecting rates

Suppose the spot rate is 5% and the forward rate one year out is 6%. What are the one- and two-year zero-coupon rates? What are the one- and two-year par-coupon rates?

Duration formulas

traditional (Macauley) duration:

$$duration = \sum_{t-1}^{T} \frac{c_t D(0,t)}{\Sigma_{s=1}^{T} c_s D(0,s)} \; t$$

The discount factors $D(0,t)$ are usually computed using either the bond's yield (i.e. $D(0,t) \equiv 1/(1+y)^t$) or using the actual discount factors. Macauley duration assumes that random shocks impact forward rates equally at all maturities.

effective duration (sens $=$ short for sensitivity):

$$sens(\textbf{effective duration}) = \left(\frac{\Sigma_{s=1}^{T} sens(s) c_s D(0,s)}{\Sigma_{s=1}^{T} c_s D(0,s)} \right)$$

For effective duration, shocks affect different forward rates differently, so the amount of interest rate exposure is no longer proportional to time-to-maturity, even for a discount bond.

the particular effective duration measure we have used:

$$sens(duration) = \exp(-.125 * duration)$$
$$duration(sens) = -\log(1 - .125 * sens)/.125$$

In-class exercise: duration and effective duration

Suppose the yield curve today is flat at 5%. Compute the duration and effective duration of a portfolio paying $100, 10 years from now, and $100, 20 years from now. Recompute the duration and effective duration assuming a flat yield curve at 10%.

Option pricing formulas

single-period:

$$\text{Value} = R^{-1}(\pi_U V_U + \pi_D V_D)$$

expected present value computed using artificial "risk-neutral" probabilities... Risk-neutral probabilities could be computed from the payoffs of some asset, but more commonly we make assumptions about them directly.

multi-period:

$$\text{Value} = E^* \left[\frac{1}{R_1} \frac{1}{R_2} \frac{1}{R_3} ... \frac{1}{R_T} C_T \right]$$

This formula is especially useful for simulations but can also be used in simple binomial cases without American or conversion features.

In-class exercise: binomial model

The spot interest rate is 5%. Each year it goes up by 5% (e.g. from 5% to 10%) with risk-neutral probability $1/3$ or down by 2% (e.g. from 5% to 3%) with risk-neutral probability $2/3$. What is the price of a 2-year interest-rate cap with a capped rate of 5% and underlying notional amount of $\$100$ trillion?

Mean reversion and fudge factors

For mean reversion

$$E[r_{t+1} - r_t] = k(\overline{r} - r_t)$$

in the binomial model with equal changes δ or $-\delta$ in rates, set

$$\pi_U = \frac{1}{2} + \frac{k(\overline{r} - r_t)}{2\delta}$$

Without mean reversion, $k = 0$ and $\pi_U = 1/2$.

fudge factors: To match actual discount factors $D(0, t)$, modify the original model—om—as follows:

$$R_s = R_s^{om} \frac{D(0, s - 1)/D(0, s)}{D^{om}(0, s - 1)/D^{om}(0, s)}$$

or approximately

$$r_s = r_s^{om} + f(0, s) - f^{om}(0, s)$$

In-class exercise: capstone problem with fudge factors and mean reversion

Consider a two-year binomial model. Start with an original model in which the short riskless interest rate starts at 5% and moves up or down by 2.5% each period (i.e., up to 7.5% or down to 2.5% at the first change). The artificial probability of each of the two states at any node is determined by whatever makes mean reversion k equal to 20% per year with a long-term mean of 5%.

What is the price of a one-year discount bond in this original model? the two-year discount bond?

Suppose the actual one-year discount rate in the economy is 6% and the actual two-year discount rate is 6.5%. Compute the fudge factors and draw the tree for the adjusted interest rate process.