Problem Set 1: Kuhn-Tucker Conditions and Binomial Portfolio Optimization
Financial Optimization
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1. Optimal portfolio choice in the binomial model

Recall that in the simplest standard binomial model, in each period the riskless asset pays off $R > 1$ per dollar invested and the stock, the risky asset, pays off $U > R$ in the good state and $D < R$ in the bad state. Valuation can be done using the risk-neutral probabilities $\pi_U^* = (R - D)/(U - D)$ and $\pi_D^* = 1 - \pi_U^* = (U - R)/(U - D)$ while preferences are based on actual probabilities π_U and $\pi_D = 1 - \pi_U$. We take U, D, R, and π_U to be the same at all nodes, and we assume there is a positive risk premium so that $\pi_U > \pi_U^*$.

A state of nature is characterized by the sequence of up and down moves. All of the variables we are interested in will be path independent, so we will think of a collapsed tree and consider a final state characterized by the number of ups n and number of downs $N - n$, where N is the total number of periods. Then the final stock price is

$$S_N(n) = S_0 U^n D^{N-n}.$$

Having $n \in \{0, 1, ..., N\}$ up moves over N periods has risk-neutral probability

$$\pi^*(n, N) = \left(\begin{array}{c} N \\ n \end{array} \right) {\pi_U^*}^n {\pi_D^*}^{N-n},$$

and actual probability

$$\pi(n, N) = \left(\begin{array}{c} N \\ n \end{array} \right) {\pi_U}^n {\pi_D}^{N-n},$$

where

$$\left(\begin{array}{c} N \\ n \end{array} \right) \equiv \frac{N!}{n!(N-n)!}$$

is the *binomial coefficient* giving the number of paths with n up moves over N periods.

Assume a von Neumann-Morgenstern (expected) utility function u of consumption $c > 0$ of the form

$$u(c) = \begin{cases} \log(c) & \text{if } \gamma = 1 \\ \frac{c^{1-\gamma}}{1-\gamma} & \text{otherwise} \end{cases}$$

where $\gamma > 0$. These are called CRRA (constant relative risk aversion[1]) or isoelastic preferences. For all CRRA preferences, $u'(c) = c^{-\gamma}$.

To keep things simple, suppose $N = 2$. Assume that $U = 2$, $D = 1$, and $R = 5/4$. Assume further that $\pi_U = \pi_D = 1/2$, $\gamma = 1$ (log case), and $W_0 = 36$.

A. Write down the optimization problem for maximizing expected utility of terminal consumption subject to the budget constraint that the expected present value of terminal consumption under the risk-neutral probabilities is equal to the initial wealth W_0.

Choose c_0, c_1, and c_2 to
maximize $\pi(0,2)\log(c_0) + \pi(1,2)\log(c_1) + \pi(2,2)\log(c_2)$
subject to $\pi^*(0,2)c_0 + \pi^*(1,2)c_1 + \pi^*(2,2)c_2 = W_0$

B. Write down the Kuhn-Tucker conditions for the optimization problem.

$$\nabla f = \left(\frac{\pi(0,2)}{c_0} + \frac{\pi(1,2)}{c_1} + \frac{\pi(2,2)}{c_2} \right)$$

$$\nabla g = \frac{1}{R^2} \left(\frac{\pi^*(0,2)}{c_0} + \frac{\pi^*(1,2)}{c_1} + \frac{\pi^*(2,2)}{c_2} \right)$$

Since we only have an equality constraint, there are no complementarity slackness conditions, and the Kuhn-Tucker conditions are $\nabla f(c^*) = \lambda \nabla g(c^*)$

[1] This comes from the Arrow-Pratt measure $-u''(c)/u'(c)$ of relative risk aversion. These utility functions have the same preference for relative (proportional) gambles independent of the wealth level.

plus the budget constraint. So, we have

$$(\forall n = 0,1,2)\frac{\pi(n,2)}{c_n} = \lambda\frac{\pi^*(n,2)}{R^2}$$

$$\pi^*(0,2)c_0 + \pi^*(1,2)c_1 + \pi^*(2,2)c_2 = W_0$$

C. Solve the optimization problem. (You do not need to confirm the second-order conditions, but they do hold.)

Solving the first-order conditions we have, for $n = 0,1,2$, that

$$c_n = \frac{R^2}{\lambda}\frac{\pi(n,2)}{\pi^*(n,2)}.$$

Plugging into the budget constraint, we have

$$\frac{1}{R^2}\left(\pi^*(0,2)\frac{R^2}{\lambda}\frac{\pi(0,2)}{\pi^*(0,2)} + \pi^*(1,2)\frac{R^2}{\lambda}\frac{\pi(1,2)}{\pi^*(1,2)} + \pi^*(2,2)\frac{R^2}{\lambda}\frac{\pi(2,2)}{\pi^*(2,2)}\right) = W_0$$

$$\frac{1}{\lambda}(\pi(0,2) + \pi(1,2) + \pi(2,2)) = W_0$$

Since the probabilities sum to 1, this says that $1/\lambda = W_0$ and therefore

$$c_n = \frac{\pi(n,2)}{\pi^*(n,2)}W_0R^2$$

Recall that $\pi_U = \pi_D = 1/2$ and therefore

$$\begin{aligned}\pi(0,2) &= \frac{2!}{0!2!}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^2 \\ &= \frac{2}{1\times 2}\frac{1}{4} = \frac{1}{4},\end{aligned}$$

$$\begin{aligned}\pi(1,2) &= \frac{2!}{1!1!}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^1 \\ &= \frac{2}{1\times 1}\frac{1}{4} = \frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\pi(2,2) &= \frac{2!}{2!0!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 \\ &= \frac{2}{2\times 1}\frac{1}{4} = \frac{1}{4}.\end{aligned}$$

Now $\pi^*_U = (R-D)/(U-D) = (1/4)/1 = 1/4$, and $\pi^*_D = (U-R)/(U-D) = (3/4)/1 = 3/4$. Therefore we have

$$\begin{aligned}\pi^*(0,2) &= \frac{2!}{0!2!}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^2 \\ &= \frac{2}{1\times 2}\frac{9}{16} = \frac{9}{16},\end{aligned}$$

$$\begin{aligned}\pi^*(1,2) &= \frac{2!}{1!1!}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^1 \\ &= \frac{2}{1\times 1}\frac{3}{16} = \frac{3}{8},\end{aligned}$$

and

$$\begin{aligned}\pi^*(2,2) &= \frac{2!}{2!0!}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^0 \\ &= \frac{2}{2\times 1}\frac{1}{16} = \frac{1}{16},\end{aligned}$$

So, we have

$$\begin{aligned}c_0 &= \frac{\pi(0,2)}{\pi^*(0,2)}R^2W_0 \\ &= \frac{1/4}{9/16}(5/4)^2 36 \\ &= 25,\end{aligned}$$

$$
\begin{aligned}
c_1 &= \frac{\pi(1,2)}{\pi^*(1,2)} R^2 W_0 \\
&= \frac{1/2}{3/8}(5/4)^2 36 \\
&= 75,
\end{aligned}
$$

$$
\begin{aligned}
c_2 &= \frac{\pi(2,2)}{\pi^*(2,2)} R^2 W_0 \\
&= \frac{1/4}{1/16}(5/4)^2 36 \\
&= 225.
\end{aligned}
$$

which is the solution.

D. (extra for experts) Compute the dynamic portfolio strategy to follow to get the optimal payoff at the end.

The wealth at all points in the tree can be computed as:

$$
\begin{aligned}
W_{UU} &= c_2 = 225 \\
W_{UD} &= c_1 = 75 \\
W_{DD} &= c_0 = 25 \\
W_U &= \frac{1}{R}(225\pi_U^* + 75\pi_D^*) = \frac{4}{5}\left(\frac{1}{4}225 + \frac{3}{4}75\right) = 90 \\
W_D &= \frac{1}{R}(75\pi_U^* + 25\pi_D^*) = \frac{4}{5}\left(\frac{1}{4}75 + \frac{3}{4}25\right) = 30 \\
W_0 &= \frac{1}{R}(90\pi_U^* + 30\pi_D^*) = \frac{4}{5}\left(\frac{1}{4}90 + \frac{3}{4}30\right) = 36
\end{aligned}
$$

(the last one we already knew but it is good to check)

Portfolio weights are computed using replicating portfolios. Initially, we have

$$2S + \frac{5}{4}B = 90$$

$$S + \frac{5}{4}B = 30$$

Subtract the equations to get $S = 60$ and plug into one equation to get $B = -24$. $S + B = 36$, as it should. After an initial up move, we solve

$$2S + \frac{5}{4}B = 225$$

$$S + \frac{5}{4}B = 75$$

to obtain $S = 150$ and $B = -60$, so $S + B = 90$, as it should. After an initial down move, we solve

$$2S + \frac{5}{4}B = 75$$

$$S + \frac{5}{4}B = 25$$

to obtain $S = 50$ and $B = -20$, so $S + B = 30$, as it should.

To summarize, the portfolio is initially long 60 worth of stock and short 24 worth of bond. If the stock goes up, we switch to long 150 worth of stock and short 60 worth of bond, while if instead the stock goes down, we switch to long 50 worth of stock and short 20 worth of bond.

2. Portfolio Insurance

A. Write down the same problem as in problem 1 above but with the additional constraint that terminal consumption is no smaller than initial wealth.

B. Write down the Kuhn-Tucker conditions for this problem.

C. Solve the optimization problem under the same assumptions about parameters as in part C of Problem 1.

D. (extra for experts) Compute the dynamic portfolio strategy to follow to get the optimal payoff at the end.

3. (challenger) Solve the problem in part 1 or 3 above using the utility function

$u(c) = \max(1, 2\sqrt{c})$.

Warning: this is a nonconvex problem and you cannot just solve the first-order conditions.