Problem Set 2 Answers: Linear Programming
P. Dybvig

At the start of class next week, submit only problem 3 for grading. For your study, these are answers to the other questions (except the challenger).

1. Standard form

Convert the following LPs to standard form. Be sure to explain the relation between the variables in the new and old problems.

Choose $x_1 \geq 0$ and $x_2 \geq 0$ to
maximize $x_1 + x_2$, subject to
$x_1 \leq 6$ and $x_1 + 2x_2 \leq 3$.

The choice variables are already nonnegative, so that is already standard. The problem is a maximization problem so we have to change it to a minimization problem by flipping the sign. And we need to change the inequality constraints to equality constraints by introducing slack variables, call them z_1 and z_2. Now the choice problem is

Choose nonnegative x_1, x_2, z_1, and z_2 to
minimize $-x_1 - x_2$, subject to
$x_1 + z_1 = 6$ and
$x_2 + 2x_2 + z_2 = 6$.

In matrix-and-vector notation (using y instead of x used in the slides to avoid a conflict in notation), this is

Choose $y \geq 0$ to
minimize $c^\top y$
subject to $Ay = b$

where $y^\top = (x_1, x_2, z_1, z_2)$, $c^\top = (-1, -1)$, $b^\top = (6, 6)$, and

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}.$$

The original choice variables are $x_1 = y_1$ and $x_2 = y_2$.

Choose x_1 and x_2 to
maximize $2x_1 + x_2$, subject to
$x_2 - x_1 \leq 1$, $x_1 - x_2 \leq 1$, $x_2 + x_1 \geq -1$, and $-x_1 - x_2 \geq -1$.

The choice variables are not restricted to be nonnegative, so we need new variables to allow for that possibility. Let $x_1 = z_1 - z_2$ and $x_2 = z_3 - z_4$ where z_1,...,z_4 are nonnegative. We also have the flip the sign to make this a minimization. We also add new slack variables z_5, ... z_8 to make the inequalities into equalities. With these changes, the problem becomes

Choose z_1, z_2, ... z_8 to minimize $-2z_1 + z_2 - z_3 + z_4$, subject to $z_3 - z_4 - z_1 + z_2 + z_5 = 1$, $z_1 - z_2 - z_3 + z_4 + z_6 = 1$, $z_3 - z_4 + z_1 - z_2 - z_7 = -1$, and $-z_1 + z_2 - z_3 + z_4 - z_8 = -1$.

in matrix-and-vector notation, this becomes

Choose $z \geq 0$ to
minimize $c^\top z$
subject to $Az = b$

where $z^\top = (z_1, z_2, ..., z_8)$, $c^\top = (-2, -1)$, $b^\top = (1, 1, -1, -1)$, and

$$A = \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 * -1 & \end{pmatrix}.$$

The original choice variables are $x_1 = z_1 - z_2$ and $x_2 = z_3 - z_4$.

2. Asset-Liability Application and Lagrange Multiplier

Modify the pension fund example from class (spreadsheet available on the class page) to include an additional bond paying 14 each in years 1 through 8.

A. First, use the Lagrange multipliers to show that if the bond costs 100 initially it is too expensive and will not be held.

The Lagrange multipliers (shadow prices) for the cash flow constraints given by Solver are

$$(0.9690, 0.9353, 0.8993, 0.8614, 0.8219, 0.7812, 0.7398, 0.6978)$$

The objective function is in terms of cash at the start, and relaxing a constraint is an injection of cash at some future time. Therefore, the impact on the objective function of buying the new bond is -100 (the immediate impact on cash) plus 0.9690×14 (the increase in initial cash from gettin 14 a year from now) plus 0.9353×14 (the increase in initial cash from gettin 14 two years from now), etc. Adding up the terms, we have the NPV of the bond using the shadow prices as discount factors:

$$-100 + 0.9690 \times 14 + 0.9353 \times 14 + 0.8993 \times 14 + 0.8614 \times 14 + 0.8219 \times 14 + 0.7812 \times 14 + 0.7398 \times 14 + 0.6978 \times 14 = -6.1202.$$

Because the value is negative, it is not attractive to buy the bond. In general, we do not know for how large a trade this is a good approximation, but we do know that this is an upper bound for how good it is (because this is a convex optimization problem).

B. Verify this by adding the bond with price 100 in solver and checking that the solution is unchanged.

There is more than one way to do this. I inserted a new row at 20 and I put "bond 17" in cell A20 and 0 in cell B20. (If you insert a different number of rows or you do this in a different order, your addresses may be different from mine below.) This set up the label and input cell for the choice variable giving the new bond holding. Then I put "bond 17" in cell A42, 100 in cell B42, and 14 in all cells C42:J42. This set up the label and all the cash flows for the new bond. Then I took the formula SUMPRODUCT($B$4:$B$19,C26:C41) in cell C24 to include the new bond: SUMPRODUCT($B$4:$B$20,C26:C42), and then I copied across to C24:J24. At this point, I looked around at these formulas to check to make sure I did what I intended, and I realized that I did not adjust the objective function to include the new bond. Then, I changed cell E6 from SUMPRODUCT(B5:B19,B27:B41) SUMPRODUCT(B5:B20,B27:B42).

Next, I changed the solver settings to reflect the new set-up. In Solver (entered in the Tools menu in my copy of Excel), the target cell is still correct as are the constraints. However, there is a new choice variable, so we change "By Changing Cells" from $B$4:$B$19 to $B$4:$B$20. There is nothing to change in the Options menu, since we still want "Assume Linear Model" and "Assume Non-Negative." So, I press solve and ask for the "Answer" report which confirms that the solution has not changed (if unsure, check against the answers in the slides).

C. Try instead adding the bond with price 90 and verify that the solution changes.

I start with the spreadsheet in part B and I change cell B42 to 90. In solver, I press solve, and I use Shift-click to select both answer and sensitivity reports.

The values for the choice variables I get are:

0,0,0.02,0,0.05064,0,0.08205792,0,0.1244376,0,0.138419603,0,0.112433771,0, 0.085694351,0,4.867698

and the new value for the objective is 496.0729368.

D. Describe in words the original solution and how the solution changes when the new bond is introduced. (This should be like the description you would give a boss who wants a verbal description, not all the quantitative details.)

Before introducing the new bond, the solution used one bond each at each maturity, sometimes using the bond with the higher coupon, sometimes using the bond with the lower coupon. With the new bond available, cost is down $19 million and the new bond accounts for almost all the value (about 98%) because it is much cheaper than the other bonds and its constant cash flow are about the right pattern to match the liabilities. The remaining small investment goes into the bonds maturing in years 2 through 8 with the smaller yields.