FIN 550 Practice Exam Answers

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1. True-False (25 points)

A. Linear programs typically have interior solutions.

False. Unless the objective is zero, all solutions are at the boundary.

B. A local optimum of a convex optimization problem is a global optimum.

True.

C. The Fundamental Theorem of Asset Pricing is about the absence of arbitrage.

True.

D. The product of the eigenvalues equals the trace of a matrix.

False. The trace is the sum of the eigenvalues. The determinant is the product of the eigenvalues.

F. Slack variables are used to convert inequality constraints into equality constraints.

True.

2. Linear Programming (25 points)

Consider the following linear program:

Choose nonnegative x_1, x_2, and x_3 to
maximize $2x_1 + x_2 + 3x_3$, subject to
$x_1 + 2x_2 + 3x_3 \leq 6$ and
$x_1 + x_2 \leq 3$

A. What is the dual linear program?

Choose nonnegative y_1 and y_2 to
minimize $6y_1 + 3y_2$, subject to
$y_1 + y_2 \geq 2$,
$2y_1 + y_2 \geq 1$, and
$3y_1 \geq 3$.

B. Is the primal feasible? Is the dual feasible?

Yes to both. $(0, 0, 0)$ is feasible in the primal and $(1, 1)$ in the dual.

C. Infer from the answers in part B: Is the primal bounded? Is the dual bounded?

Yes to both, because primal feasible iff dual bounded and dual feasible iff primal bounded.

D. Solve the dual problem.

The second constraint is redundant. We know from parts B and C there is a solution and from first principles that the solution has to be at an extreme point of the feasible set. Graphing the constraints shows that extreme points in the feasible set are $(1, 1)$ and $(2, 0)$; $(1, 1)$ is the solution because it has the smaller value $9 < 12$.

E. Use the solution to the dual problem to solve the primal problem.

Since the middle constraint in the dual is not binding at the solution, its shadow price is zero so that $x_2 = 0$ in the solution to the primal. Also, y_1 and y_3 are not zero so the shadow prices of the first and third constraints in the primal are positive and these constraints must be binding. Solving $x_1 + 3x_3 = 6$ and $x_1 + 0 = 3$, we have $x_1 = 3$ and $x_3 = 1$. Therefore the solution is $(x_1, x_2, x_3) = (3, 0, 1)$. This has value 9, the same as the value of the dual. Since we have found feasible solutions to the primal and the dual with the same value, they must be solutions to the problems, giving an independent check on the analysis.

3. REGIME-SWITCHING (25 points) Consider a two-state Markov Chain in continuous time. Regime switches take place at the following rates:

$$\begin{array}{ll} \text{state } 1 \to \text{state } 2 & \text{probability } 0.1/\text{year} \\ \text{state } 2 \to \text{state } 1 & \text{probability } 0.05/\text{year} \end{array}$$

Initially (at time $t = 0$), we are in state 2.

a. What is the matrix A in the ODE

$$\pi'(t) = A\pi(t)$$

describing the dynamics of the vector $\pi(t)$ of future regime probabilities?

The off-diagonal entry A_{ij} is the probability of moving to state i given that we are now in state j, and the on-diagonal entry A_{ii} makes column i sum to 0. Therefore,

$$A = \begin{pmatrix} -.1 & .05 \\ .1 & -.05 \end{pmatrix}.$$

b. Solve for the eigenvalues of A.

$$\begin{aligned} 0 &= \det(A - \lambda I) = \det \begin{pmatrix} -.1 - \lambda & .05 \\ .1 & -.05 - \lambda \end{pmatrix}. \\ &= (-.1 - \lambda)(-.05 - \lambda) - .1 \times .05 \\ &= \lambda^2 + .15\lambda + .005 - .005 = \lambda(\lambda + .15) \end{aligned}$$

Therefore, $\lambda_1 = 0$ and $\lambda_2 = -.15$ are the eigenvalues.

c. Solve for the associated eigenvectors.

$$(A - \lambda_1 I)q = Aq = A = \begin{pmatrix} -.1 & .05 \\ .1 & -.05 \end{pmatrix} q = 0.$$

We can take $q^2 = 1$ and then $q^1 = 1/2$ so we can take $q^1 = (1/2, 1)^T$ (or any nonzero scalar multiple of this vector).

$$(A - \lambda_2 I)q = A = \begin{pmatrix} .05 & .05 \\ .1 & .1 \end{pmatrix} q = 0.$$

We can take $q^2 = 1$ and then $q^1 = -1$ so we can take $q^2 = (-1, 1)^T$ (or any nonzero scalar multiple of this vector).

d. Write down the general solution of the ODE.

Since the ODE is homogeneous, the homogeneous solution is the general solution:

$$\begin{aligned} \pi(t) &= K_1 e^{\lambda_1 t} q^1 + K_2 e^{\lambda_2 t} q^2 \\ &= K_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + K_2 e^{-.15t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned}$$

e. Write down the particular solution corresponding to the initial condition that we start in state 2.

This condition is $y(0) = (0, 1)^T$ which implies that

$$K_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution to this is $K_1 = 2/3$ and $K_2 = 1/3$. Therefore, the specific solution is

$$\pi(t) = \begin{pmatrix} \frac{1-e^{-.15t}}{3} \\ \frac{2+e^{-.15t}}{3} \end{pmatrix}$$

f. A project costing \$90,000 has a cash flow of \$6,000/year in state 2 and \$1,000/year in state 1. If the interest rate is 5%/year, does this project have a positive NPV?

In thousands of dollars, we have cash flow of $c = (1, 6)^T$/year in the two states and we can write the present value as

$$PV = \int_{t=0}^{\infty} e^{-.05t} c^T \pi(t) dt$$

$$
\begin{aligned}
&= \int_{t=0}^{\infty} e^{-.05t}\left(1\frac{1-e^{-.15t}}{3}+6\frac{2+e^{-.15t}}{3}\right)\\
&= \frac{13}{3}\int_{t=0}^{\infty} e^{-.05t}dt+\frac{5}{3}\int_{t=0}^{\infty} e^{-.20t}dt\\
&= \frac{13}{3\times 1/20}+\frac{5}{3\times 1/5}\\
&= \frac{260+25}{3}=95
\end{aligned}
$$

Therefore, the NPV is $95-90=5$ or $\$5,000$.

4. Kuhn-Tucker Conditions (25 points)

Consider the following optimization problem:

Choose c_u and c_d to
maximize $\frac{1}{2}\log(c_u)+\frac{1}{2}\log(c_d)$, subject to
$\frac{4}{5}(\frac{1}{4}c_u+\frac{3}{4}c_d)\leq 6$.

This is a single-period choice of investment for consumption in a binomial model with log utility, initial wealth of 6, actual probabilities 1/2 and 1/2, risk-neutral probabilities 1/4 and 3/4, and riskfree rate of 25% (and therefore discount factor 4/5).

A. What are the objective function, choice variables, and constraint?

objective function: $\frac{1}{2}\log(c_u)+\frac{1}{2}\log(c_d)$
choice variables: c_u and c_d
constraint: $\frac{4}{5}(\frac{1}{4}c_u+\frac{3}{4}c_d)\leq 6$

B. What are the Kuhn-Tucker conditions?

$(\frac{1}{2c_u},\frac{1}{2c_d})=\lambda(\frac{1}{5},\frac{3}{5})$, $\lambda\geq 0$, and
$\lambda(\frac{4}{5}(\frac{1}{4}c_u+\frac{3}{4}c_d)-6)=0$

C. If we add constraints $c_u\geq 6$ and $c_d\geq 6$, what are the Kuhn-Tucker conditions now?

$(\frac{1}{2c_u}, \frac{1}{2c_d}) = \lambda(\frac{1}{5}, \frac{3}{5}) + \lambda_u(-1, 0) + \lambda_d(0, -1)$,
λ, λ_u, and $\lambda_d \geq 0$, $\lambda(\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) - 6) = 0$,
$\lambda_u(6 - c_u) = 0$, and
$\lambda_d(6 - c_d) = 0$.

5. Bonus question (30 bonus points)

A. Solve the optimization problem in problem 4 without the extra constraints in part 4C.

There is more than one way of solving this problem and the next one.

$\frac{1}{2c_u} = \lambda\frac{1}{5} \Rightarrow c_u = \frac{5}{2\lambda}$
$\frac{1}{2c_d} = \lambda\frac{3}{5} \Rightarrow c_d = \frac{5}{6\lambda}$
substituting into the budget constraint, which must be binding or else we could increase the objective function by increasing consumption:: $\frac{1}{5}\frac{5}{2\lambda} + \frac{3}{5}\frac{5}{6\lambda} = 6 \Rightarrow \lambda = \frac{1}{6}$. therefore: $c_u = 15$ and $c_d = 5$ which can easily be verified to satisfy the K-T conditions for $\lambda = 1/6$. Since this is a convex optimization with a strictly concave objective function and the KT conditions are not degenerate (the gradients of binding g's are linearly independent), this is the solution.

B. Solve the optimization problem in problem 4 with the extra constraints in part 4C.

We know the budget constraint is binding since otherwise increasing consumption will increase value. Since the constraint $c_d \geq 6$ is violated in the solution in part A and $c_u \geq 6$ is satisfied and not binding, it is reasonable to conjecture that $c_d \geq 6$ is the only new constraint binding in the solution here.[1] Then the budget constraint implies $c_u = 12$. To prove this is a solution, we solve for the multipliers. Since the constraint $c_u \geq 6$ is not binding, the c. slackness condition $\lambda_u(6 - c_u) = 0$ implies $\lambda_u = 0$ and therefore

[1] Of course, this can be derived by going through cases of what constraints are binding. If we assume both new constraints are binding, $c_u = c_d = 6$, which does not satisfy the budget constraint. If we assume neither new constraint is binding, we obtain the solution in part A, which is not feasible. If we assume the new constraint $c_u \geq 6$ is binding but the other new constraint is not, we find that $\lambda_u < 0$, which is not consistent with the first-order conditions.

$\frac{1}{2c_u} = \lambda\frac{1}{5} \Rightarrow \lambda = \frac{5}{24}$.
Then we can also solve for λ_d:
$\frac{1}{2c_d} = \lambda\frac{3}{5} - \lambda_d \Rightarrow \lambda_d = \lambda\frac{3}{5} - \frac{1}{2c_d} = \frac{5}{24}\frac{3}{5} - \frac{1}{12} = \frac{1}{24}$.
It is easy to verify that the KT conditions are satisfied by choosing $c_u = 12$, $c_d = 6$, $\lambda = 5/24$, $\lambda_u = 0$, and $\lambda_d = 1/24$. Since this is a nondegenerate solution of the KT conditions for a convex optimization with a strictly concave maximand, $(c_u, c_d) = (12, 6)$ is the unique solution.