FIN 550 Exam Answers

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1. True-False (25 points)

A. Constrained problems never have interior solutions.

FALSE

B. In an unconstrained problem with a concave objective function, a local maximum is a global maximum.

TRUE

C. Multiplying a random variable by a constant greater than 1 increases its kurtosis.

FALSE

D. The eigenvalues of a positive definite matrix are all positive.

TRUE

E. A linear program with a bounded dual always has an optimal solution.

FALSE

2. Linear Programming (25 points) Consider the following linear program:

Choose nonnegative $x_1$, $x_2$, and $x_3$ to
maximize $x_1 + 6x_2 + 3x_3$, subject to
$x_1 + 3x_2 + 3x_3 \leq 6$ and
$x_1 + x_2 \leq 3$

A. What is the dual linear program?

Choose nonnegative $y_1$ and $y_2$ to
minimize $6y_1 + 3y_2$, subject to

$$\begin{array}{lcl} y_1 + y_2 & \geq & 1 \\ 3y_1 + y_2 & \geq & 6 \\ 3y_1 & \geq & 3 \end{array}$$

B. Solve the dual problem.

Plotting the constraints shows that the feasible set has corners $(2,0)^T$ and $(1,3)^T$. Therefore the problem is feasible and it is also bounded because the objective function is nonnegative for all feasible $y$ (which are nonnegative), and therefore either $(2,0)^T$ or $(1,3)^T$ (or both) must be a solution. The objective function at $(2,0)^T$ is $6 \times 2 + 3 \times 0 = 12$ and the objective function at $(1,3)^T$ is $6 \times 1 + 3 \times 3 = 15$. Therefore, $(2,0)^T$ is the solution.

C. Use the solution to the dual problem to solve the primal problem.

In the dual problem, at the solution $(2,0)^T$, only the second constraint holds with equality and therefore the multipliers for the first and third constraints, which are also the optimal the choice variables $x_1$ and $x_3$ for the primal problem, must be zero. Therefore, we have $x_1 = x_3 = 0$. Also, $y_1 = 2 > 0$ is the optimal choice for the first variable in the dual problem and this is therefore the multiplier for the first constraint in the primal, which must be binding by complementarity slackness. Given the first constraint in the primal is binding and $x_1 = x_3 = 0$, we have that $x = (0,2,0)^T$ is the solution to the primal problem.

D. Show that the strong duality theorem holds in this example.

The value of the solution $x = (0,2,0)^T$ of the primal problem is $0 + 6 \times 2 + 3 \times 0 = 12$, which we have already seen in part B to be the value of the solution $(2,0)^T$ of the dual problem. Therefore, the strong duality theorem, which says the primal and dual problems have the same value, is satisfied.

3. REGIME-SWITCHING (25 points) Consider a two-state Markov Chain in continuous time. Regime switches take place at the following rates:

$$\begin{array}{ll} \text{state } 1 \rightarrow \text{state } 2 & \text{probability } 0.10/\text{year} \\ \text{state } 2 \rightarrow \text{state } 1 & \text{probability } 0.15/\text{year} \end{array}$$

Initially (at time $t = 0$), we are in state 2.

A. What is the matrix $A$ in the ODE

$$\pi'(t) = A\pi(t)$$

describing the dynamics of the vector $\pi(t)$ of future regime probabilities?

$$\begin{bmatrix} -.1 & .15 \\ .1 & -.15 \end{bmatrix}$$

B. Solve for the eigenvalues of $A$.

Since the eigenvalues are the solutions of $\det(A - \lambda I) = 0$, we have that

$$0 = \det \begin{bmatrix} -.1 - \lambda & .15 \\ .1 & -.15 - \lambda \end{bmatrix} = \lambda^2 + .25\lambda = \lambda(\lambda - (-.25))$$

and therefore the eigenvalues are $\lambda_0 = 0$ and $\lambda_1 = -.25$.

C. Solve for the associated eigenvectors.

We will look for a nonzero solution to $(A - \lambda I)x = 0$. Since eigenvectors are determined only up to nonzero scale, we arbitrarily set $x_2 = 1$ (although we will change this choice if needed). So, for $\lambda_0 = 0$, we have

$$\begin{bmatrix} -.1 & .15 \\ .1 & -.15 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = 0$$

so $x_1 = 1.5$ and we can take the first eigenvector to be $x^0 = (1.5, 1)^T$.

For $\lambda_1 = -.25$, we have

$$\begin{bmatrix} .15 & .15 \\ .1 & .1 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \end{bmatrix} = 0$$

so $x_1 = -1$ and we can take the second eigenvector to be $x^1 = (-1, 1)^T$.

D. Write down the general solution of the ODE.

$$
\begin{aligned}
\pi(t) &= a_0 e^{\lambda_0 t} x^0 + a_1 e^{\lambda_1 t} x^1 \\
&= a_0 \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} + a_1 e^{-.25t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\end{aligned}
$$

E. Write down the particular solution corresponding to the initial condition that we start in state 2.

Substituting $\pi(0) = (0,1)^T$, we have $a_0 = .4$ and $a_1 = .6$, so that

$$
\pi(t) = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + e^{-.25t} \begin{bmatrix} -.6 \\ .6 \end{bmatrix}.
$$

F. A project costing $\$100,000$ has a cash flow of $\$10,000$/year in state 1 and $\$1,000$/year in state 2. The cash flows continue forever. If the continuously-compounded interest rate is 5%/year, does this project have a positive expected NPV?

Define $c = (10,1)^T$ to be the vector of costs across states, in thousands. Then we have

$$
\begin{aligned}
E[PV] &= \int_{t=0}^{\infty} \left( c^T \left( \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} -.6 \\ .6 \end{bmatrix} e^{-.25t} \right) \right) e^{-.05t} dt \\
&= \frac{6.4}{.05} - \frac{5.4}{.30} \\
&= 128 - 18 = 110.
\end{aligned}
$$

Therefore, the NPV is $110 - 100 = 10$ and the project does have a positive expected NPV.

4. Kuhn-Tucker Conditions (25 points)

Consider the following optimization problem:

Choose $c_u$ and $c_d$ to
maximize $\frac{2}{3}\log(c_u) + \frac{1}{3}\log(c_d)$, subject to
$\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$,
$c_d \geq 6$,
and
$c_u \geq 6$.

This is a single-period choice of investment for consumption in a binomial model with log utility, initial wealth of 6, actual probabilities 2/3 and 1/3, risk-neutral probabilities 1/2 and 1/2, and riskfree rate of 25% (and therefore discount factor 4/5).

A. What are the objective function, choice variables, and constraints?

objective function: $\frac{2}{3}\log(c_u) + \frac{1}{3}\log(c_d)$
choice variables: $c_u$ and $c_d$
constraints: $\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$, $c_d \geq 6$, and $c_u \geq 6$.

B. What are the Kuhn-Tucker conditions?

$$\begin{pmatrix} \frac{2}{3c_u} \\ \frac{1}{3c_d} \end{pmatrix} = \lambda \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix} + \lambda_u \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_d \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad (*)$$
$\lambda_u(6 - c_u) = 0$ and $\lambda_d(6 - c_d) = 0$
$\lambda_u \geq 0$ and $\lambda_d \geq 0$

optionally include the constraints:

$\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$
$c_d \geq 6$
$c_u \geq 6$

C. Solve the optimization problem.

Since the left-hand side of the gradient equation (*) is positive and $\lambda_u$ and $\lambda_d$ are nonnegative, $\lambda > 0$. If we have $\lambda_u = \lambda_d = 0$, then solving the budget constraint and the gradient equation (*) we obtain $c_u = 10$ and $c_d = 5$, which violates the constraint $c_d \geq 6$. Therefore, it is natural to conjecture that constraint is binding so that $c_d = 6$ and from the budget constraint

$c_u = 9$. This satisfies all the constraints and the Kuhn-Tucker conditions if we set $\lambda = 1/6$, $\lambda_u = 0$, and $\lambda_d = 1/15 - 1/18 > 0$. (It is easy to compute $\lambda_d = 1/90$, but we don't actually need to calculated it since $1/15 - 1/18$ is obviously positive.