Problem Set 1: Probability, answer to Problem 2
FIN 550: Numerical Methods and Optimization in Finance
P. Dybvig

2. Assume the stock price S three months from now has an exponential distribution with scale parameter $\theta > 0$, i.e. the density of S is

$$f(S) = \begin{cases} \frac{1}{\theta} e^{-S/\theta} & \text{for } S \geq 0 \\ 0 & \text{for } S < 0 \end{cases}$$

and the cumulative distribution function of S is

$$F(S) = \begin{cases} 1 - e^{-S/\theta} & \text{for } S \geq 0 \\ 0 & \text{for } S < 0 \end{cases}$$

Consider a call option on this stock maturing three months from now with a strike price $X > 0$. The payoff of the call option is

$$C = \max(S - X, 0).$$

A. What is the cumulative distribution function of the option payoff?

$$\begin{aligned} G(C) &= \text{prob}(\max(S - X, 0) < C) \\ &= \begin{cases} \text{prob}(S < X + C) & \text{for } C \geq 0 \\ 0 & \text{for } C < 0 \end{cases} \\ &= \begin{cases} F(X + C) & \text{for } C \geq 0 \\ 0 & \text{for } C < 0 \end{cases} \\ &= \begin{cases} 1 - e^{-(X+C)/\theta} & \text{for } C \geq 0 \\ 0 & \text{for } C < 0 \end{cases} \end{aligned}$$

We can see that C is never negative, $C = 0$ with probability $1 - e^{-X/\theta}$, and $C > 0$ with positive density $g(C) = G'(C) = (1/\theta)e^{-(X+C)/\theta}$.

B. What is the expected option payoff?

$$E[C] = (1 - e^{-X/\theta})0 + \int_{C=0}^{\infty} \frac{1}{\theta} e^{-(X+C)/\theta} C dC$$

$$
\begin{aligned}
&= 0 + e^{-X/\theta} \int_{C=0}^{\infty} \frac{1}{\theta} e^{-C/\theta} C dC \\
&\text{now integrated by parts: } U = C \text{ and } V = -e^{-C/\theta} \\
&\text{(recall } \textstyle\int U dV = [UV] - \int V dU\text{)} \\
&= e^{-X/\theta} \left(\left[-Ce^{-C/\theta} \right]_0^{\infty} - \int_{C=0}^{\infty} -e^{-C/\theta} dC \right) \\
&= e^{-X/\theta} \left(0 + \left[-\theta e^{-C/\theta} \right]_0^{\infty} \right) \\
&= \theta e^{-X/\theta}
\end{aligned}
$$

Note that if $X = 0$, this equals θ, which is the mean of the exponential distribution, as it should because the call is the same as the stock when $X = 0$.

C. What is the variance of the option payoff?

$$
\begin{aligned}
E[C^2] &= (1 - e^{-X/\theta})0^2 + \int_{C=0}^{\infty} \frac{1}{\theta} e^{-(X+C)/\theta} C^2 dC \\
&= 0 + e^{-X/\theta} \int_{C=0}^{\infty} \frac{1}{\theta} e^{-C/\theta} C^2 dC \\
&\text{now integrate by parts: } U = C^2 \text{ and } V = -e^{-C/\theta} \\
&= e^{-X/\theta} \left(\left[C^2 e^{-C/\theta} \right]_0^{\infty} - \int_{C=0}^{\infty} -e^{-C/\theta} 2C dC \right) \\
&= 2\theta e^{-X/\theta} \int_{C=0}^{\infty} \frac{1}{\theta} e^{-C/\theta} C dC \\
&\text{In part B, we already computed the integral} = \theta \\
&= 2\theta^2 e^{-X/\theta}
\end{aligned}
$$

Therefore,

$$
\text{var}(C) = E[C^2] - (E[C])^2 = 2\theta^2 e^{-X/\theta} - \theta^2 e^{-2X/\theta}
$$

Note that if $X = 0$, this equals θ^2, which is the variance of the exponential distribution, as is should because the call is the same as the stock when $X = 0$.