Problem Set 2: Optimization, answer to Problem 1
FIN 550: Numerical Methods and Optimization in Finance
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1. Consider the following maximization problem:

Choose c_1 and c_2 to
maximize $3\log(c_1) + 2\log(c_2)$
subject to:
  $c_1 + c_2 \leq 100$
and
  $40 \leq c_2$

A. Write the constraints in the appropriate form for the Kuhn-Tucker conditions. Be careful to get the signs correct for the constraint functions.

For a maximization problem, we want inequality constraints in the form of $g_i(c) \leq 0$. Therefore, we have two constraints $g_1(c) \leq 0$ where $g_1(c) = c_1 + c_2 - 100$ and $g_2(c) \leq 0$ where $g_2(c) = 40 - c_2$ (not $c_2 - 40$, which would have the inequality backwards).

Note that there are implicit inequality constraints $c_i > 0$ since this is where the objective function is defined. However, we do not have to include these constraints explicitly since the boundary where $c_i = 0$ for some i is not feasible.

B. Compute the gradient of the objective function and the gradient of each of the constraint functions.

Denote the objective function by $f(c) = 3\log(c_1) + 2\log(c_2)$. Then we have

$$\nabla f(c) = \left(\frac{3}{c_1}, \frac{2}{c_2}\right)^T.$$

For the constraints, we have $\nabla g_1(c) = (1,1)^T$ and $\nabla g_2(c) = (0,-1)^T$.

C. Write down the Kuhn-Tucker conditions.

$$\begin{pmatrix} \frac{3}{c_1} \\ \frac{2}{c_2} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$
$$(c_1 + c_2 - 100)\lambda_1 = 0$$
$$(40 - c_2)\lambda_2 = 0$$
$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$
$$c_1 + c_2 - 100 \leq 0$$
$$40 - c_2 \leq 0$$

Note: I have included the constraints of the problem in the K-T conditions. This is optional.

D. Solve the problem.

If we conjecture that no constraints are binding so $\lambda_1 = \lambda_2 = 0$, then we have $3/c_1 = 0$ and $2/c_2 = 0$ which cannot be (we are tempted to write $c_1 = c_2 = +\infty$ but that makes no sense in this setting and would not satisfy the budget constraint).

If we conjecture that both constraints are binding, solving the two constraints for c_1 and c_2 gives $c_2 = 40$ (from the second constraint) and then $c_1 = 60$ (from the first constraint). Then we can solve the K-T condition

$$\begin{pmatrix} \frac{3}{60} \\ \frac{2}{40} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

to obtain $\lambda_1 = 1/20$ and $\lambda_2 = 0$. Although the second constraint is not binding (in the strict sense that the multiplier is not zero), this is still a solution satisfying all the K-T conditions. The constraints and complementary slackness conditions follow because the constraints are satisfied with equality, λ_1 and λ_2 are nonnegative, and gradient equation is easy to verify (and holds because this is how we chose the λ's).

E. Prove that the solution is correct. (Probably, you will want to prove that a solution to the Kuhn-Tucker conditions with positive c_1, and c_2 satisfying the constraints of the problem must be an optimal solution.)

The second derivative of the objective function is

$$f''(c) = \begin{pmatrix} -\frac{3}{{c_1}^2} & 0 \\ 0 & -\frac{2}{{c_2}^2} \end{pmatrix}$$

which is negative definite everywhere in the domain $\Re_{++} \times \Re_{++}$ of the objective function, since the eigenvalues are $-\frac{3}{{c_1}^2}$ (with associated eigenvector $(1,0)^T$) and $-\frac{2}{{c_2}^2}$ (with associated eigenvector $(0,1)^T$). Therefore, the objective function is strictly concave. Furthermore, the feasible set is convex: since the constraints are all linear, the feasible set is convex because is the intersection of half-planes that are convex. Since the objective function is concave and the feasible set is convex, any solution of the Kuhn-Tucker conditions (including the original constraints) is an optimal solution of the problem (which in our case must be unique since the objective function is strictly concave). Our claimed solution satisfies all the Kuhn-Tucker conditions (including the constraints), so it must be optimal.