FINANCIAL OPTIMIZATION

Lecture 3: LP Duality, Fundamental Theorem of Asset Pricing

Philip H. Dybvig
Washington University
Saint Louis, Missouri

Dual LP

For the LP (called the primal)

Choose $x \geq 0$ to
minimize $c^\top x$
subject to $Ax \geq b$

there is a dual LP

Choose $y \geq 0$ to
maximize $b^\top y$
subject to $A^\top y \leq c$

Intuition: y is a vector of shadow prices, one for each constraint in $Ax \geq b$. If the original LP chooses quantities to maximize revenue, the dual LP chooses prices to minimize cost.

Note: the original LP is the dual of the dual (flipping signs as needed).

recall the convention: $x \geq 0 \Leftrightarrow (\forall i) x_i \geq 0$

In-Class Exercise: Dual Problem

Write down the dual to the following LP:

Choose nonnegative x_1, x_2, and x_3 to
minimize $x_1 + x_2 + 2x_3$, subject to
$x_1 \geq 3$,
$2x_2 + 2x_3 \geq 4$
$x_2 + 3x_3 \geq 4$.

Dual Problem as a Best Bound

Suppose y is feasible in the dual. Then $A^\top y \le c$ or equivalently $y^\top A \le c^\top$. For any feasible $x \ge 0$ in the primal, we have that $Ax \ge b$. Combining these results, we therefore have that the objective function evaluated at x in the primal is no smaller than the objective function evaluated at y in the dual:

$$\begin{aligned} c^\top x &\ge y^\top A x \\ &\ge y^\top b. \end{aligned}$$

In other words, any feasible solution to the dual has a value that is no larger than any feasible solution to the primal. This is the weak duality theorem given on the next page. This also implies that the value of the primal is no smaller than the value of the dual, and in fact it can be proven they are equal if they both exist and are finite. That is the strong duality theorem. Another theorem, whose name I do not know, shows the connection between feasibility in the primal and boundedness of the dual, and *vice versa*.

Duality theorems

Weak duality theorem: if x is feasible in the primal and y is feasible in the dual, then $c^\top x \geq b^\top y$.

Strong duality theorem: if x^* is optimal in the primal and y^* is optimal in its dual, then $c^\top x^* = b^\top y^*$, i.e., the two programs have the same value. Also, the vector of shadow prices in the primal is the vector of optimal choices in the dual and *vice versa*.

For the final result, we want to use a special definition of bounded that does not assume feasibility. The primal (resp. dual) is called unbounded if there is a direction Δ of improvement $c^\top \Delta < 0$ (resp. $b^\top \Delta > 0$) in which feasibility is not impaired: $\Delta \geq 0$ and $A\Delta = 0$ (resp $A^\top \Delta = 0$). Using this definition:

The LP is feasible if and only if the dual is bounded. The dual is feasible if and only if the LP is bounded.

In the book, a program is only called unbounded if it is feasible, which gives a less useful result in one direction only (because if the primal is infeasible the dual may be infeasible as well).

In-class Exercise: Strong Duality

(1) In the linear program in the previous in-class exercise, show that $x = (3, 1, 1)$ is feasible in the primal and $y = (1, 1/4, 1/2)$ is feasible in the dual.

(2) What are the values of the two problems for these choices?

(3) Write down an optimal solution to the primal and an optimal solution to the dual.

Strong Duality: equivalence of Kuhn-Tucker Conditions

primal choice problem	dual choice problem
Choose $x \geq 0$ to minimize $c^\top x$ subject to $Ax \geq b$	Choose $y \geq 0$ to maximize $b^\top y$ subject to $A^\top y \leq c$

primal first-order conditions	dual first-order conditions
$c^\top = \lambda A + \lambda^x$ $(\forall i) x_i \lambda_i^x = 0$ $(\forall j)(Ax - b)_j \lambda_j = 0$ $Ax \geq b$ λ^x, λ, $x \geq 0$	$b^\top = \gamma A^\top + \gamma^y$ $(\forall j) y_j \gamma_j^y = 0$ $(\forall i)(A^\top y - c)_i \gamma_i = 0$ $A^\top y \leq c$ γ^y, γ, $y \geq 0$

Variable correspondence: $\lambda = y$, $\lambda^x = c - A^\top y$, $x = \gamma$, and $\gamma^y = Ax - b$

Some Practical Uses of the Duality Theorems

- Verify a claimed solution (as in the In-class Exercise)
- Some models can be solved more quickly in the dual
- Derive qualitative properties (no more than two activities used)
- Quick bounds on the value can be derived by hand

Example (book Exercise 2.11)

The following problem can be solved graphically in the dual (only two choice variables) and then the primal variables can be inferred using complementary slackness.

Choose x_1, x_2, x_3, x_4, x_5 to
maximize $6x_1 + 5x_2 + 4x_3 + 5x_4 + 6x_6$, subject to
$x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$ and
$5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \leq 14$

Fundamental Theorem of Asset Pricing

I was happy to see the Fundamental Theorem of Asset Pricing in the book, since I invented this term and contributed to the theory. However, I am disappointed with the way it is handled in the book so I will give you my own approach. I first came up with the term when I was teaching an important paper by Steve Ross,[1] and I was looking for a good way to explain the result. I came up with what I first called "The Fundamental Theorem of Finance" but then I decided that was too grandiose and that fewer people would object to "The Fundamental Theorem of Asset Pricing." This result relates asset pricing, absence of arbitrage, and portfolio choice. Together with the related "Pricing Rule Representation Theorem," it is a conceptual foundation for much of what we do in mathematical finance.

[1] Ross, Stephen A., 1978, A Simple Approach to the Valuation of Risky Streams, *Journal of Business* **51**, 453–475.

For More Information

A good simple exposition of the theory can be found in my article with Steve: "Arbitrage," an entry in the New Palgrave: A Dictionary of Economics. This entry was originally published in 1987 and also appears with minor changes in the 2008 edition of the dictionary. The dictionary is available online through the Washington Universities library site:

http://library.wustl.edu/databases/about/pdoe.html

Like many other resources in the library, this needs to be accessed from a university IP address or (what amounts to the same thing) using a university proxy server. The exposition in class is based mostly on that dictionary entry.

Verbal Statements

Fundamental Theorem of Asset Pricing The following are equivalent:

(i) absence of arbitrage
(ii) existence of a positive linear pricing rule
(iii) existence of an optimal demand for some agent who prefers more to less

Pricing Rule Representation Theorem The following are equivalent:

(i) existence of a positive linear pricing rule
(ii) existence of positive risk-neutral probabilities and an associated riskfree rate (the martingale property)
(iii) existence of a positive state-price density

Prices and Payoffs

We will do everything in a single-period model with finitely many states. The results are true more generally, but if we work with continuously many states or infinitely many points of time, some additional regularity must be assumed.

- p vector of asset prices. could include zeros (futures) or even negative numbers (for investments without limited liability)
- X state-space tableau, $X_{\theta i}$ is the payoff to asset i in state θ
- Arbitrage η such that $(-p^\top \eta, X\eta) > 0$
- Vector of exogenous state probabilities $\pi >> 0$
- Consistent linear pricing rule: $q >> 0$ such that $p = qX$.
- Risk-neutral valuation $r > -1$ and $\pi^* >> 0$ such that $p = \pi^* X/(1+r)$ and $\Sigma_\theta\, \pi^*_\theta = 1$.
- Positive state-price density: $\rho >> 0$ such that $(\forall i) p_i = \Sigma_\theta (\pi_\theta \rho_\theta) X_{\theta i}$

Notation: $x \geq 0 \Leftrightarrow (\forall i) x_i \geq 0$; $x > 0 \Leftrightarrow x \geq 0$ and $x \neq 0$; $x >> 0 \Leftrightarrow (\forall i) x_i > 0$

Ideas behind the proofs

Fundamental Theorem of Asset Pricing:

(i)⇒(ii) This is basically a separation theorem or extension theorem. The article uses the theorem of the alternative, which is sometimes used to prove the strong duality theorem of linear programming. The book proves it using a result from linear programming.

(ii)⇒(iii) In the article, this is proven by construction.

(iii)⇒(i) Immediate, because an agent cannot have an optimum if there is an arbitrage (since adding the arbitrage would make the agent better off.

Pricing Rule Representation Theorem:

(i)⇒(ii) Let $r_f = 1/(\Sigma_\theta\, p_\theta) - 1$, and let $\pi^* = (1 + r)p$.

(ii)⇒(iii) Let $\rho_\theta = \pi^*_\theta/((1 + r)\pi_\theta)$.

(iii)⇒(1) let $p_\theta = \pi_\theta \rho_\theta$.

Uses of Pricing Rule Representations

a rough guide to when to use which one:

(i) General linear pricing rule $L(x)$ or $\Sigma_\theta\, p_\theta x_\theta$: proving no-arbitrage theorems such as Modigliani-Miller or put-call parity.

(ii) Risk-neutral pricing: valuation exercises when probabilities do not matter, for example, most option-pricing exercises.

(iii) State-price density (or stochastic discount factor): optimal portfolio choice problems in which probabilities do matter, especially for time-separable von Neumann-Morgenstern preferences for which marginal utilities are proportional to the state-price density.

Practical Use of the No-Arbitrage Approach

The no-arbitrage approach of Ross is a great tool for organizing your thinking about security markets, and it is also very useful in practice. In its simplest form, the tool can be used to look for an arbitrage in a frictionless market. A more elaborate version can look for riskless or near-riskless profit opportunities in the presence of such frictions as bid-ask spread, trading costs, price pressure, taxes, and restrictions on leverage or short selling.

If we have unrestricted holdings, the condition that we never lose money is a simple inequality constraint:

$$\begin{pmatrix} -p^\top \\ X \end{pmatrix} \eta \geq 0.$$

Note that each column of the array is a feasible net trade, and we are treating cash up front just like any state tomorrow (or another day, or in a different currency or commodity). What is slightly trickier is to include the possibility of an arbitrage profit, since this is defined by having a strict inequality in some contingency. Also, having an arb would make any reasonable optimization unbounded: what to do?

An Optimization to Find an Arbitrage

Choose an unrestricted vector η to
maximize $\mathbf{1}^\top Y\eta$
subject to $Y\eta \geq 0$ and
$\mathbf{1}^\top Y\eta \leq 1.$

where

$$Y \equiv \begin{pmatrix} -p^\top \\ X \end{pmatrix}$$

is the state-space tableau for all net trades, inclusive of cash up-front, and $\mathbf{1}$ is a vector of 1's.

In-Class Exercise: Finding Arbitrage

binomial model: let $U = 2$, $D = 1/2$, $R = 5/4$, initial stock price 50, and an at-the-money call on the stock costs 15.

1. What are the state-space tableau X and corresponding price vector p?
2. What is an optimization problem that can be used to find an arbitrage?

Defining the Discrete Tableau in a Continuous World

- Discretization: valuation on a grid
- Piece-wise linear: use breakpoints
- Spline fit: use critical set of points
- Polynomial fit: match coefficients

None of these is perfect and some common sense is required!

An Optimization to Find an Arbitrage: Bid-Ask Spread

Choose nonnegative vector η to
maximize $\mathbf{1}^\top Z\eta$
subject to $Z\eta \geq 0$ and
$\mathbf{1}^\top Z\eta \leq 1.$

where $\eta^\top = (\eta_L^\top, \eta_S^\top)$ includes first long and then short positions in all the assets, and

$$Z \equiv \begin{pmatrix} -p_A^\top & p_B^\top \\ X & -X \end{pmatrix}$$

is the state-space tableau for all net trades, inclusive of cash up-front, separating long positions entered at the ask prices p_A and short positions entered at the bid prices p_B.