INVESTMENTS
Lecture 3: Mean-variance theory and the CAPM

Philip H. Dybvig
Washington University in Saint Louis

- Decision theory
- Mean-variance theory
- Means, variances, and covariances
- The value of diversification
- Optimal portfolio choice

Decision Theory

Decision Theory gives us a conceptual framework for formalizing optimal choice. This framework is almost indispensible if we want to solve mathematicaally for an optimal choice, and it is also useful for thinking about a choice problem even if we are going to use a combination of intuition and informal analysis to make the final decision.

The essential pieces of a choice problem are the *choice variables*, the *objective function*, and the *constraints*. We may also have *parameters*, which are inputs to the choice problem that can be varied.

A Simple "Consumer's" Decision Problem

Choose c_1, c_2, and c_3 to
maximize $U(c_1, c_2, c_3)$ subject to
$p_1c_1 + p_2c_2 + p_3c_3 = W$,
$c_1 \geq 0$,
$c_2 \geq 0$, and
$c_3 \geq 0$.

The *choice variables* are c_1, c_2, and c_3, which are expenditures on three classes of consumption goods. The *objective function* is $U(c_1, c_2, c_3)$, which represents the consumer's preferences for different consumption patterns. There are four *constraints*, the *budget constraint* and three *positivity constraints*. Some *parameters* of the problem include the prices p_1, p_2, and p_3, and wealth W. There may be additional parameters describing the features of the utility function. The parameters are not chosen as part of the decision; we leave them flexible to allow us to study, for example, the sensitivity of the consumption of good one to its price.

A Simple Investments Problem

Choose q_1, q_2, and q_3 to
maximize $E[u(w_1)]$ subject to
$q_1 + q_2 + q_3 = 1$ and
$w_1 = w_0(q_1(1 + r_1) + q_2(1 + r_2) + q_3(1 + r_3))$.

The *choice variables* are q_1, q_2, and q_3, which are proportional investments in three different securities. The *objective function* is $E[u(w_1)]$, which represents the investor's preferences over different random payoffs. The first *constraint* is the budget constraint that proportions in the securities sum to one, and the second constraint defines the payoff w_1 given the amounts invested and the random returns r_1, r_2, and r_3 on the three securities. To complete this specification, we would have to specify the particular utility function u and the joint probability distribution of the returns.

An Even Simpler Investments Problem

Suppose there are two assets, a riskless asset 1 with return 10% and a risky asset 2 with equally probable returns -10% and +50%. Assuming that the utility function is $u(w_1) = w_1 - .004{w_1}^2$ and the initial wealth is 100, we can substitute in the constraints (e.g. $q_1 = 100\% - q_2$) and use elementary algebra (see the following slide) to reduce the previous sort of choice problem to the following.

Choose q_2 to
maximize $61.6 + 1.2q_2 - 4.0{q_2}^2$.

The solution is $q_2 = .15$, i.e. 15% of wealth should be in the risky asset and 85% in the riskless asset. (One way of proving this is by writing the objective function as $61.69 - 4.0(q_2 - 0.15)^2$.) This is a very conservative strategy; a less conservative solution would arise if we replaced .004 with a smaller number, representing a smaller aversion to risk-taking.

An Even Simpler Investments Problem (Algebra Details)

$$
\begin{aligned}
E[u(w_1)] &= \frac{1}{2}u(w_{1u}) + \frac{1}{2}u(w_{1d}) \\
&= 0.5(u(100(1.1q_1 + 1.5q_2)) + u(100(1.1q_1 + .9q_2))) \\
&= 0.5(u(100(1.1(1-q_2) + 1.5q_2)) + u(100(1.1(1-q_2) + .9q_2))) \\
&= 0.5(u(110 + 40q_2) + u(110 - 20q_2)) \\
&= 0.5\big((110 + 40q_2) - .004(110 + 40q_2)^2 \\
&\quad + (110 - 20q_2) - .004(110 - 20q_2)^2\big) \\
&= 0.5\big(110 + 40q_2 - 48.4 - 35.2q_2 - 6.4q_2^2 \\
&\quad + 110 - 20q_2 - 48.4 + 17.6q_2 - 1.6q_2^2\big) \\
&= 61.6 + 1.2q_2 - 4q_2^2 \\
&= 61.69 - 4(.15 - q_2)^2 \\
&\leq 61.69
\end{aligned}
$$

(since the square cannot be negative), which is the value when $q_2 = 0$.

Types of portfolio problems

For an individual's portfolio choice, the choice problem in the previous slides is a good start. In most institutional settings, there are at least two levels of management. At the highest level, the plan sponsor (the representative of the beneficiaries) must choose proportions of the portfolio to be allocated to different asset classes and more specifically how to allocate funds within each asset class to different managers (who may or may not be in-house). We call this selection of broad asset classes *asset allocation*. The problem of the specific manager (who may manage a portfolio of equities or government bonds or convertible bonds), we refer to as subportfolio management.

Traditionally, academic finance has not looked at subportfolio problems, which are significantly different from asset allocation problems. Subportfolio managers are typically judged relative a benchmark appropriate to the asset class and are constrained directly and indirectly in how far they can deviate from the benchmark. The traditional tools for asset allocation can be modified in a natural way to study subportfolio management.

Mean-variance theory

Mean-variance theory is an important model of investments based on decision theory. It is the simplest model of investments that is sufficiently rich to be directly useful in applied problems. Mean-variance theory was developed in the 50's and 60's by Markowitz, Tobin, Sharpe, and Lintner, among others. Ironically, it is still called Modern Portfolio Theory (MPT) by some people. While no longer the most modern model, mean-variance theory continues to be the main workhorse on which analytical portfolio management is based. The equilibrium version of mean-variance theory is called the Capital Asset Pricing Model (CAPM).

The nicest feature of mean-variance theory is its simplicity. By assuming that preferences depend only on the mean and variance of payoffs and not on other features, we obtain a number of robust results.

Mean-variance theory and the CAPM: main ideas

- Take on risk in proportion to the risk premium and in inverse proportion with variance and risk aversion.
- Diversification pays.
- The market rewards you for taking a share of economy-wide risk.
- The market does not reward you for taking on security-specific (idiosyncratic) risk.
- All investors hold a mixture of two portfolios, one riskless (if there is a riskless asset), and the market portfolio.

The assumptions of mean-variance theory

For the simple decision problem, the assumptions are:

- Single-period model
- Preferences depend only on the mean and variance of payoffs
  - At a given mean, lower variance is preferred
  - At a given variance, a higher mean is preferred
- Price-taking with no taxes or transaction costs

For the equilibrium model (CAPM): we have the above assumptions and

- no information asymmetry
- competitive equilibrium

The assumptions of no taxes, transaction costs, or information asymmetry are sometimes known collectively as the assumption of "perfect capital markets."

Population and sample statistics

For these statistics, there are "population" versions (which is what you would expect or what you would see in a hypothetical large sample comprised of the entire population of possible events) and "sample" versions telling what actually happened. For example, if we have a coin we believe to be fair, the population probability of "heads" is $1/2$. If we have seen 1000 flips of this coin, 508 heads and 492 tails, the sample probability of "heads" is $508/1000 = 0.508$. We will define various statistics in terms of their sample versions, but it is important to keep in mind the difference between sample and population values. In many contexts, the sample version is a good estimate of the population version. However, due to the amount of volatility in security returns, sample means may be very bad estimates of the population means. As a result, using the sample version in a model expecting the population version can produce bizarre prescriptions. In particular, it may instruct you to put extreme amounts of money into securities and sectors that performed better than expected in the past.

Review: means, variances, and covariances

The mean return, a measure of a typical value, is the usual arithmetic average:

$$mean = \frac{1}{n}\sum_{i-1}^{n} R_i$$

The variance, a measure of volatility or dispersion, is the average squared deviation from the mean:

$$variance = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - mean)^2$$

The covariance, a measure of co-movement, is the average product of deviations from the mean:

$$covariance^{xy} = \frac{1}{n-1}\sum_{i-1}^{n}(R_i^x - mean^x)(R_i^y - mean^y)$$

Covariances are important in portfolio theory because they tell us whether risks cancel or compound when assets are combined in portfolios.

Portfolio variances from individual asset variances

Consider two assets, 1 and 2, whose returns have variances S_1^2 and S_2^2, respectively, and whose returns have covariance S_{12}. Then the variance of a portfolio with weight W_1 in the first asset and weight W_2 in the second asset (with residual weight $1 - W_1 - W_2$ in the riskless asset) has variance

$$W_1^2 S_1^2 + 2W_1 W_2 S_{12} + W_2^2 S_2^2$$

It can be shown that $-S_1 S_2 \leq S_{12} \leq S_1 S_2$ (or equivalently the *correlation coefficient* $S_{12}/(S_1 S_2)$ must always lie between -1 and 1).

For a portfolio with many assets, there are many cross terms like the middle one here. If there are n assets, there are n variance terms and $n(n-1)/2$ cross terms. In a portfolio with a typical universe of assets, estimating all the covariances needed for the cross terms is an important practical question, since the number of covariances may exceed the number of data points.

Security returns: market returns and idiosyncratic noise.

For shares of stock, we can think of the return being equal to a mean return plus a random part from market-level noise plus a random part idiosyncratic to the firm. Mathematically, this means that

$$R_i = mean_i + \beta_i z_m + e_i,$$

where $mean_i$ and β_i are constants, and the market noise z_m and the idiosyncratic noise terms e_i are all uncorrelated (and therefore have zero covariances). This assumption that idiosyncratic risk is uncorrelated across assets is not strictly true (and multifactor models seem to fit better), but this assumption will give us the right intuition.

Single asset variance

The variance of a single asset is given by

$$\beta_i^2 var(z_m) + var(e_i).$$

For a typical large stock, we might have $\beta_i = .8$, $var(z_m) = .04$, $var(e_i) = .09$, and therefore the asset's return variance is $.1156$. In this example, the market standard deviation is 20% and the asset's standard deviation is $\sqrt{.1156} = 34\%$. For a typical small stock, we might have $\beta_i = 1.5$, $var(z_m) = .04$, and $var(e_i) = .16$, and therefore the asset's return variance is $.25$. In this example, the market standard deviation is 20% and the asset's standard deviation is 50%.

The value of diversification

As an example, assume that we put equal amounts of money into *n* assets. Then the resulting portfolio has variance

$$\left(\frac{1}{n}\sum_i \beta_i\right)^2 var(z_m) + \frac{1}{n}\left(\frac{1}{n}\sum_i var(e_i)\right)$$

As the portfolio gets larger, the idiosyncratic risk term becomes less and less important, and we can approximate the portfolio's risk by the first term.

Orders of magnitude

Following on the previous slide, suppose we form portfolios with assets all having the same $\beta = 1$ and isiosyncratic standard deviation $.3$. Suppose further that the market standard deviation is $.2$. Then using the formula on the previous slide, we find that the standard deviation of a portfolio is computed as $\sqrt{.2^2 + .3^2/n}$. Here are some values of this function:

number of assets	standard deviation
1	36.06%
2	29.15%
5	24.08%
10	22.14%
20	21.10%
50	20.45%
100	20.22%
1000	20.02%

Although not all assets have the same idiosyncratic variance, this still gives an accurate qualitative picture of the value of diversification. Note that diversification does not affect the mean return, which will be the mean of the individual asset mean returns.

Optimal portfolio choice: a simple problem

We will not go through the algebra of optimal portfolio choice. The main message I take from the algebra is that we tilt into each uncorrelated source of risk in proportion to the expected return we get and in inverse proportion to variance and risk aversion. This basic rule tells how to take consistent exposures to different risks.

As a first simple problem, suppose our optimal mix when choosing only between riskless asset and the market portfolio is *50%* in each, where the riskless rate is *5%*, the mean return on the market is *15%*, and the standard deviation of the market is *20%*. Suppose then that we have a stock "Island" that has a beta of zero (this means all of its risk is uncorrelated with the market), a mean return of *10%*, and a standard deviation of *50%*. How much of our portfolio should we invest in each of Island, the market, and the riskless asset?

Solution of the simple problem

We know that the investment in uncorrelated sources of risk should be in proportion to their mean excess return over variance. In the case of the market, the mean excess return over variance is *(15%-5%)/.04=2.5* and this justifies a commitment of *50%* of the portfolio. In the case of Island stock, we have mean excess return over variance of *(10%-5%)/.25=0.2*. Since this is proportional to the commitment, we should commit *50%(0.2/2.5)=4%* of the portfolio to Island stock. After committing half to the market portfolio and *4%* to Island stock, we have left *100%-50%-4%=46%* to invest in the riskless asset.

Sophisticated problem with correlated returns

Our example of Island Corporation stock was made particularly simple because of the beta of zero which meant that no manipulation was required to treat uncorrelated sources of risk. In general, we should regroup the assets into portfolios that provide "pure plays" in uncorrelated sources of risk.

Assume the market portfolio and base problem are as in the previous example, i.e., that our optimal mix of riskless asset and the market portfolio is *50%* in each, where the riskless rate is *5%*, the mean return on the market is *15%*, and the standard deviation of the market is *20%*. Suppose then that we have a stock "Hitek" with a beta of *1.5*, a mean return of *30%*, and an idiosyncratic standard deviation of *50%*. What is our optimal holding of Hitek, the market portfolio, and the riskless asset?

Sophisticated problem: solution

In the problem, we are not given our level of risk aversion, but we will take that to be implicit in our optimal mix between the market and the riskless asset. In that case, the mean excess return over variance *10%/.04=2.5* induced us to invest half our wealth. To use this in the new problem, we need to focus on uncorrelated risk, that is a net position with the market risk removed. In this case, consider an investment that is long *100%* share of Hitek, short *150%* in the market, and long *150%* in the riskless asset. This new investment has a beta of *0* (it is uncorrelated with the market as we wish), a mean return of *15%*, and a standard deviation of *50%*. The mean excess return of this portfolio is *30% - (1.5)15% + (1.5)5% - 5% = 10%*. Therefore, the mean excess return over variance of this is *10%/.25=.4*, and we should invest a proportion of wealth *(.4/2.5)50% = 8%* in this strategy. This implies overall holdings of *50% - (1.5)8% = 38%* in the market portfolio, *8%* in Hitek stock, and *100% - 38% - 8% = 54%* in the riskless asset.

Optimal portfolio choice: in-class exercise

Suppose our optimal mix of riskless asset and the market portfolio is *50%* in each, where the riskless rate is *5%*, the mean return on the market is *15%*, and the standard deviation of the market is *20%*. Suppose then that we favor a stock "Bloochip" with a beta of *1.0*, a mean return of *20%*, and an idiosyncratic standard deviation of *25%*. What is our optimal holding of Bloochip, the market portfolio, and the riskless asset?