INVESTMENTS
Lecture 4: Dynamic Models

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- Fixed proportion portfolio strategies
- Log-optimal policy
- Review of binomial option pricing

Why Dynamic Models?

It should perhaps be obvious that investments in actuality span many short periods. In general, there will be connections across the periods. If there is consumption withdrawal, then there is the question of how much to take out for use this period and how much to leave in the portfolio for future use. Even when we are accumulating to a fixed horizon (for example, building wealth that will be paid out in a lump sum at retirement), there is a question of whether investment proportions should vary depending on time or investment performance. Finally, even given fixed proportions, we want to learn what our choices about asset proportions imply about the distribution of wealth available at the end.

Fixed investment proportions

We are going to start with the simplest dynamic model, in which we take investment proportions as fixed. We will take very simple assumptions; volatilities and expected returns are known in advance and do not vary over time. Literally speaking, these are not the most accurate assumptions we could make, but the results we obtain will not mislead us.

Multiperiod returns

If we start with initial wealth W_0 and invest in a portfolio whose rate of return in each period t is R_t, then the wealth after T periods is given by:

$$W_T = W_0 \prod_{t=1}^{T} (1 + R_t).$$

In statistics, we are more familiar with dealing with sums than with products, which we can do by taking logarithms. (We could use any base for the logarithms, but for convenience we will use base e. This is convenient because, for R not too large, $\log(1 + R) \approx R$ when the logarithm is in base e.)

$$log(W_T) = log(W_0) + \sum_{t=1}^{T} log(1 + R_t).$$

The distribution of multiperiod returns: fixed proportions

Now, let us assume the returns in each period are independent and have the same distribution. Then, if R_t has mean μ and variance V, $log(1 + R_t)$ has a mean of approximately $\mu - V/2$ and a variance of V. (Take my word or use a Taylor series expansion.) Therefore, in a sense I am not making precise, $log(W_T)$ has an approximate normal distribution with mean $log(W_0) + (\mu - V/2)T$ and variance VT. In fact, these expressions are exact in the continuous-time model used in the very successful Black-Scholes formula, so this is likely to be a very reasonable approximation over short time intervals. Using historical averages for mean and variance will tend to overstate average performance and understate uncertainty, since we are after all studying the US stock market because it has been more successful than average. Also, assuming variance is constant may tend to understate volatility. Nonetheless, these calculations are reasonable guesses and are much better than, for example, just assuming the mean return will hold for sure!

Stock Market vs. T-Bills

What is the probability that the stock market will outperform T-Bills at various horizons? The previous slide allows us to give a reasonable answer. Take the riskfree rate to be 5%, the mean stock market return to be 15%, and the market standard deviation to be 25%, all on an annual basis. Then the analysis of the previous slide tells us that for investing in the market, $log(W_T/W_0)$ has mean $(15\% - .03125)T$ and standard deviation $\sqrt{.0625T}$. Using the same analysis says that for rolling over T-Bills, $log(W_T/W_0)$ has mean $5\%T$ and standard deviation 0. By the properties of a normal distribution, we have that the probability that the market will outperform T-Bills over a given period of length T is the same as the probability that a unit normal random variable is bigger than $-(15\% - .03125 - 5\%)T/\sqrt{.0625T}$, which can be obtained from standard tables. This is tabulated on the following slide.

Probability that the market outperforms Treasuries

# years	Treasuries	risky asset $log(W_T/W_0)$		$prob(Mkt > Trs)$
	$log(W_T/W_0)$	$mean$	std	
1/12	.0042	.0099	.0722	.532
1/2	.0250	.0594	.1768	.577
1	.0500	.1188	.2500	.608
3	.1500	.3562	.4330	.683
5	.2500	.5938	.5590	.731
10	.5000	1.1875	.7906	.808
30	1.5000	3.5625	1.3693	.934

This table is based on historical numbers being good predictors of future returns. Personally, I think those numbers overstate reasonable market returns because we are studying a successful market because it was successful. Even given these optimistic numbers, it is a bad idea to take as a sure thing that the realized returns will be bigger than we would get in Treasuries, even over decades!

Log-optimal policy

The *log-optimal* policy is the strategy that maximizes the expected log of terminal wealth, and is also the strategy that maximizes the long-term growth rate of the portfolio's value. Some people have argued that this is the best portfolio for anyone with a long horizon, since this strategy will eventually outperform any other strategy. The problem with this reasoning is that we might have to wait hundreds or thousands of years for the superior performance. In the meantime, the performance could be disastrous.

Among fixed-proportion strategies, the expected log of terminal wealth is given by $E[log(W_T)] = log(W_0) + (r + w_1(\mu - r) - w_1^2 s^2/2)T$, where r is the riskfree rate, μ is the mean return on the market, s is the standard deviation of the market, and T is the time horizon. This expected log of terminal wealth is maximized by choosing a portfolio weight $w_1 = (\mu - r)/s^2$ in the risky market portfolio. If $\mu = 15\%$, $r = 5\%$, and $s = 20\%$, this implies that $.1/.04 = 250\%$ of the portfolio should be invested in the risky asset, which is a very aggressive policy!

Probability that the log-optimal policy outperforms Treasuries

# years	Treasuries $log(W_T/W_0)$	log optimal (l.o.) $log(W_T/W_0)$ $mean$	std	$prob(l.o. > Trs)$
1/12	.0042	.0146	.1443	.529
1/2	.0250	.0875	.3536	.570
1	.0500	.1750	.5000	.599
3	.1500	.5250	.8660	.667
5	.2500	.8750	1.1180	.711
10	.5000	1.7500	1.5811	.785
30	1.5000	5.2500	2.7386	.915

There is little assurance over reasonable horizons that the log-optimal strategy will outperform Treasuries, let alone other portfolio strategies taking some advantage of the higher average returns to stocks.

A curious feature of fixed-proportions strategies

When the mean and variance of stock returns are constant (as we have been assuming), there is a very interesting property of fixed-proportion strategies that is a good approximation in practice. Let M_T be the value at time T of a policy of staying 100% invested in an indexed portfolio of stocks, and let W_T be the value of being invested $k\%$ in stocks, each given the same initial investment. Then W_T is approximately proportional to ${M_T}^k$, or more precisely

$$W_T = W_0 \left(\frac{M_T}{M_0}\right)^k exp\left(\left((1-k)r + \frac{(k-k^2)s^2}{2}\right)T\right)$$

where s^2 is the market's variance and r is the riskfree rate. This formula is exact in the ideal limit of continuous trading, as in the Black-Scholes model.

And now for something completely different: a review of binomial option pricing

Why option pricing? Option pricing theory is ideal for analyzing or devising dynamic trading strategies, since option pricing theory gives us both the value and the trading strategy for various payoff rules. Many trading strategies such as portfolio insurance and related strategies were actually motivated by option pricing models. In general, using option pricing models allows us to customize our exposure to risk by following an investment strategy that pays off an arbitrary function of the market value at the end.

The Binomial Option Pricing Model

The option pricing model of Black and Scholes revolutionized a literature previously characterized by clever but unreliable rules of thumb. The Black-Scholes model uses continuous-time stochastic process methods that interfere with understanding the simple intuition underlying these models. We will use instead the binomial option pricing model of Cox, Ross, and Rubinstein, which captures all of the economics of the continuous time model but is simple to understand and use. For option pricing problems not appropriately handled by Black-Scholes, some variant of the binomial model is the usual choice of practitioners since it is relatively easy to program, fast, and flexible.

Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979) "Option Pricing: A Simplified Approach" *Journal of Financial Economics* **7**, 229–263

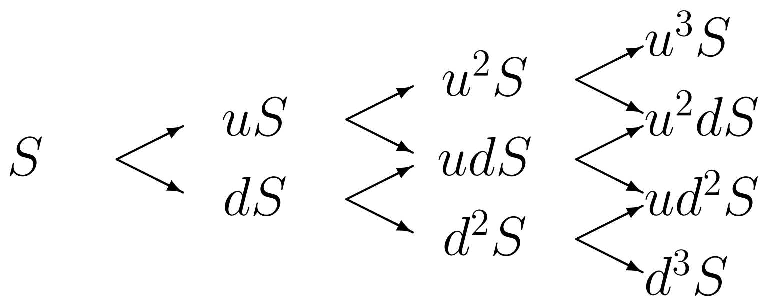
Black, Fischer, and Myron Scholes (1973) "The Pricing of Options and Corporate Liabilities" *Journal of Political Economy* **81**, 637–654

Binomial Process (3 periods)

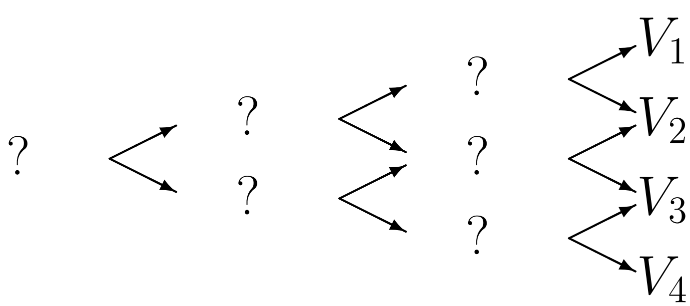
Riskless bond:

$$1 \longrightarrow r \longrightarrow r^2 \longrightarrow r^3$$

Stock ($u > r > d$):



Derivative security (option or whatever):



What is the price of the derivative security?

A Simple Option Pricing Problem in One Period

Riskless bond (interest rate is 0):

$100 \longrightarrow 100$

Stock:

$$50 \begin{cases} \nearrow 100 \\ \searrow 25 \end{cases}$$

Derivative security (on-the-money call option):

$$? \begin{cases} \nearrow 50 \\ \searrow 0 \end{cases}$$

To duplicate the call option with α_S shares of stock and α_B bonds:

$$\begin{aligned} 50 &= 100\alpha_S + 100\alpha_B \\ 0 &= 25\alpha_S + 100\alpha_B \end{aligned}$$

Therefore $\alpha_S = 2/3$, $\alpha_B = -1/6$, and the duplicating portfolio is worth $50\alpha_S + 100\alpha_B = 100/6 = 16\ 2/3$. By absence of arbitrage, this must also be the price of the call option.

In-class Exercise: One-period Contingent Claim Valuation

Compute the duplicating portfolio and the price of the general derivative security below. Assume $u > r > d > 0$.

Riskless bond:

$$1 \quad \longrightarrow r$$

Stock:

$$S \quad \begin{array}{l} \nearrow uS \\ \searrow dS \end{array}$$

Derivative security:

$$??? \quad \begin{array}{l} \nearrow V_u \\ \searrow V_d \end{array}$$

Multi-period Valuation and Artificial Probabilities

In general, exactly the same valuation procedure is used many times, taking as given the value at maturity and solving back one period of time until the beginning. This valuation can be viewed in terms of state prices p_u and p_d or risk-neutral probabilities π_u^* and π_d^*, which give the same answer (which is the only one consistent with no arbitrage):

$$Value = p_u V_u + p_d V_d = r^{-1}(\pi_u^* V_u + \pi_d^* V_d)$$

where

$$p_u = r^{-1}\frac{r-d}{u-d} \quad p_d = r^{-1}\frac{u-r}{u-d}$$

and

$$\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}.$$

In-class Exercise: Artificial Probabilities

In the binomial model (with parameters u, d, and r), show that the stock and the bond have the same one-period expected return computed using the artificial probabilities.

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 1$. What are the risk-neutral probabilities? Assuming the stock price is initially $\$100$, what is the price of a call option with a $\$90$ strike price maturing in two periods? Do the valuation two ways, using the risk neutral probabilities to solve backwards through the tree, and directly using the two-period risk neutral probabilities.

Some Orders of Magnitude

- Expected Excess Returns
  - Common stock indices: 8–10% per year or 8–$10\%/250 \approx 3$ or 4 basis points daily
  - Individual common stocks: 50%–150% of the index expected excess return.
- Standard Deviation
  - Common stock indices: 20–25**%** per year or 20–25**%**$/\sqrt{250} \approx 1$–1.5% per day
  - Individual common stocks: 35%–40% per year

Theoretical observation: for the usual case, standard deviations over short periods are almost exactly the same in actual probabilities as in risk-neutral probabilities.

Binomial Parameters in Practice

Most texts seem to have unreasonably complicated expressions for u, d, and r in binomial models. From the theory, we know that a good choice is

$$u = 1 + r * \Delta t + \sigma * \sqrt{\Delta t}$$

$$d = 1 + r * \Delta t - \sigma * \sqrt{\Delta t}$$

with $\pi_u^* = \pi_d^* = 1/2$ and Δt the time increment. This has the two essential features: it equates expected stock and bond returns, and it has the right standard deviation. In addition, it has a continuous stock price (like Black-Scholes) as a limit.

One alternative is to choose the positive solution of $ud = 1$ and $u - d = 2\sigma\sqrt{\Delta t}$ (good for option price as a function of time) or a recombining trinomial (good for including some dependence of variance on the stock price).