Problem Set 4: Optimization and Kuhn-Tucker Conditions
FIN 500J Mathematical Foundations for Finance
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Do all the calculations in the problem set by hand and show intermediate steps so the TAs will know you did not just copy the answers from the sheet with selected answers. It is okay to work together or get help from anyone else, provide you do the write-up and any computer work yourself.

The TAs will grade the homeworks.

1. Optimal investing: one period. An investor has $100,000 in wealth available for investment to retirement one year from now. The wealth can be invested in a risky asset or in a riskless bond. The payoff per dollar invested in the riskless asset is $1 in all contingencies. The payoff on the stock is random. We model the randomness by looking at two scenarios that are assumed to be equally probably; the payoff per dollar invested is $1.50 in the good scenario and $0.75 in the bad scenario.

Assume log utility, so that the optimal consumption maximizes the expectation of the log of consumption a year from now. Let θ_B be the amount of money invested in the riskless bond and let θ_S be the amount of money invested in the stock. Then the investor solves the following problem:

Choose θ_B, and θ_S to
maximize $0.5 \log(1\theta_B + 1.50\theta_S) + 0.5 \log(1\theta_B + 0.75\theta_S)$
subject to

(1) $\theta_B + \theta_S = 100000.$

a. Solve the investor's choice problem using the Kuhn-Tucker conditions.

b. Restate the investor's choice problem as an unconstrained problem by solving the budget constraint (1) for one of the two choice variables and substituting that into the objective function. Solve the unconstrained problem by setting the gradient of the objective function equal to zero, and show that the answer is the same as in part a.

2. Optimal investing: two periods. The setting is the same as in problem 1 but now the investor is two periods from retirement. In this problem, let the choice variables be the proportion of wealth ϕ_0 to invest in the risky asset at time 0 and the fraction of the risky asset $\phi_{1,s}$ to invest in the risky asset at time 1, where $s = u$ or d (up or down) is the realized state at time 0. The optimization problem is then

Choose ϕ_0, $\phi_{1,u}$, and $\phi_{1,d}$ to
maximize
$$0.25 \log(100000(1 + \phi_0(1.50 - 1))(1 + \phi_{1,u}(1.50 - 1))) +$$
$$0.25 \log(100000(1 + \phi_0(1.50 - 1))(1 + \phi_{1,u}(0.75 - 1))) +$$
$$0.25 \log(100000(1 + \phi_0(0.75 - 1))(1 + \phi_{1,d}(1.50 - 1))) +$$
$$0.25 \log(100000(1 + \phi_0(0.75 - 1))(1 + \phi_{1,d}(0.75 - 1)))$$

a. Solve this. (Hint: solve for $\phi_{1,u}$ and $\phi_{1,d}$ first and then solve for ϕ_0.)

b. (extra for experts) Prove that the portfolio proportion ϕ_0 is the same in every contingency at every point in time if there are n periods.

3. Adding constraints. We are looking at an asset allocation problem using from a mean-variance analysis. To get started, we are looking at investment in two risky asset classes with a residual investment in bonds. Let x_1 be the percentage investment in the first risky asset class and let x_2 be the percentage investment in the second risky asset class. Let $a = 100$, $b = (75, 78)^T$, and

$$C = \begin{pmatrix} 1 & .8 \\ .8 & 1 \end{pmatrix}.$$

Then the objective function will be $a + b^T x - x^T C x / 2$ where $x = (x_1, x_2)^T$. We will consider solving this optimization with and without constraints.

A. Solve the unconstrained problem:

Choose x to
maximize $a + b^T x - \frac{1}{2} x^T C x$.

B. Solve the problem with one constraint:

Choose x to
maximize $a + b^T x - \frac{1}{2} x^T C x$
subject to:
$\quad x_1 \le 40.$

C. Solve the problem with a different constraint:

Choose x to
maximize $a + b^T x - \frac{1}{2} x^T C x$
subject to:
$\quad x_2 \le 40.$

D. Solve the problem with both constraints:

Choose x to
maximize $a + b^T x - \frac{1}{2} x^T C x$
subject to:
$\quad x_1 \le 40 \quad x_2 \le 40.$