Problem Set 2: Probability using Calculus
FIN 500J Mathematical Foundations for Finance
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Prepare a hard copy of your answers to submit in class. It is okay to work with others on the problems, but do the write-up and any computer work yourself. Show your work.

1. Suppose the interest rate r a year from now has the density function

$$f(r) = \begin{cases} \frac{k}{r^4} & \text{for } r > 0.01 \\ 0 & \text{for } r < 0.01 \end{cases}$$

where k is a constant.

a. Compute the constant k.

b. Compute $E[r]$.

c. Compute $\text{var}(r)$ and $\text{std}(r)$.

2. Assume the stock price S three months from now has an exponential distribution with scale parameter $\theta > 0$ as described in the slides, i.e. the density of S is

$$f(S) = \begin{cases} \frac{1}{\theta} e^{-S/\theta} & \text{for } S > 0 \\ 0 & \text{for } S < 0 \end{cases}$$

and the cumulative distribution function of S is

$$F(S) = \begin{cases} 1 - e^{-S/\theta} & \text{for } S > 0 \\ 0 & \text{for } S < 0 \end{cases}$$

Consider a call option on this stock maturing three months from now with a strike price $X > 0$. The payoff of the call option is

$$C = \max(S - X, 0).$$

A. What is the cumulative distribution function of the option payoff?

B. What is the expected option payoff?

Hints for problem 2:

> a) This is slightly more general than the cases in class because there is a mass point with positive probability at $C = 0$ (from all the states with $S < X$) as well as a continuum of values of $C > 0$ with a density. The mean is computed by adding the probability of the mass point times the expected value of the mass point and adding the integral of the density above the mass point.
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> b) Where the density exists, the density is the derivative of the distribution function.

3. Suppose the random x and y have the joint density function

$$p(x, y) = \begin{cases} 2 - x - y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Compute $E[x]$ and $E[y]$.

b. Compute $\text{cov}(x, y)$.

c. Are x and y independent?