MATHEMATICAL FOUNDATIONS FOR FINANCE

Eigenvalues and Eigenvectors

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Background

Many problems look simple in a univariate setting but complicated in a multivariate setting. For example, consider a simple model of population

$$x_t = Ax_{t-1}.$$

If x_t is a scalar (a number, say representing the population of Missouri) and so is A, this has a simple solution and $x_t = A^t x_0$ and we know immediately what x_t is for all t. However, if x_t is a vector (say two numbers, representing the populations of Missouri and Illinois) and A is a matrix, then we can still write $x_t = A^t x_0$, but this has matrix multiplication and it is hard to see what is going on:

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ldots \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$$

For example, it is hard to see from this what is the long-term ratio of Missouri population to Illinois population.

A Special Case that Reduces to the Univariate Case

Suppose there is no population movement between Missouri and Illinois. Then the matrix A is diagonal

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

and therefore

$$\begin{aligned} A^t x_0 &= \begin{pmatrix} a_{11}^t & 0 \\ 0 & a_{22}^t \end{pmatrix} \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}^t x_{1,0} \\ a_{22}^t x_{2,0} \end{pmatrix} \end{aligned}$$

In this case, we can see exactly what is happening. The state with the higher population growth rate ($a_{11} - 1$ or $a_{22} - 1$) will have an increasing fraction of the population over time.

This case of a diagonal matrix is useful but too special for most applications.

Definition: Eigenvalues and Eigenvectors

Let A be a square matrix, and consider a scalar λ and a *nonzero* N-vector x. We say that λ is an eigenvalue of A, with associated eigenvector x, if

$$Ax = \lambda x.$$

Already we can see that this might be useful because $A^t x = \lambda^t x$, so that at least for this initial condition the dynamics are like in the univariate case.

More importantly, if λ_1, ... λ_N are eigenvalues of A with corresponding eigenvectors x_1, ... x_n, then

$$x_t = \sum_{n=1}^{N} c_n \lambda_n^t x_n$$

is the solution to $x_t = Ax_{t-1}$ with initial condition $x_0 = \Sigma_{n=1}^{N} c_n x_n$.

Computing Eigenvalues and Eigenvectors

First, solve the equation $\det(A - \lambda I) = 0$. (If $\det(A - \lambda I) \neq 0$, then there is no nonzero solution to the equation $(A - \lambda I)x = 0$, which is equivalent to the eigenvalue equation $Ax = \lambda x$.) Because of the form of the determinant, this is a polynomial equation of order N, so the roots might have to be found numerically.

Second, for each eigenvalue λ, find a nonzero solution the eigenvalue equation $Ax = \lambda x$ by solving $(A - \lambda I)x = 0$. Usually, the number of independent solutions for a given λ equals the algebraic multiplicity of λ as a root of the determinant equation above.

Computing Eigenvalues: Example

Consider

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

Now,

$$\begin{aligned} \det(A - \lambda I) &= det \begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} \\ &= (2-\lambda)(3-\lambda) - 1 \times 2 \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda - 4)(\lambda - 1) \end{aligned}$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_1 = 1$.

Computing Eigenvectors: Example

Continuing the same example with

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix},$$

for which we computed the eigenvalues to be $\lambda = 4$ and $\lambda = 1$.

For the eigenvalue $\lambda = 4$, we solve for the eigenvector by looking for a nonzero solution to the equation $(A - 4I)x = 0$, or

$$\begin{pmatrix} 2-4 & 1 \\ 2 & 3-4 \end{pmatrix} x = 0.$$

or

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} x = 0.$$

One solution of this is $x = (1, 2)^T$. Multiplying this by any nonzero scalar is also an eigenvector corresponding to the eigenvalue λ.

Computing Eigenvectors: Example (cont)

Continuing the same example with

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix},$$

for which the eigenvalues we computed to be $\lambda = 4$ and $\lambda = 1$.

For the eigenvalue $\lambda = 1$, we solve for the eigenvector for looking for a nonzero solution to the equation $(A - I)x = 0$, or

$$\begin{pmatrix} 2-1 & 1 \\ 2 & 3-1 \end{pmatrix} x = 0.$$

or

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} x = 0.$$

One solution of this is $x = (1, -1)^T$. Multiplying this by any nonzero scalar is also an eigenvector corresponding to the eigenvalue λ.

So we have that 4 is an eigenvalue with eigenvector $(1, 2)^T$ and 1 is an eigenvalue with eigenvector $(1, -1)^T$.

Population Growth: Eigenvalues and Eigenvectors

Returning to the population growth example, let

$$A = \begin{pmatrix} 1.03 & 0.005 \\ 0.02 & 1.03 \end{pmatrix}$$

and take $x_0 = (6, 13)^T$ (millions of people).

Then, A has eigenvalue $\lambda_1 = 1.04$ with corresponding eigenvector $x_1 = (1, 2)$ and eigenvalue $\lambda_2 = 1.02$ with corresponding eigenvector $x_2 = (1, -2)$.

To write the solution in terms of the eigenvalues, we first want to write the initial vector of populations as a sum of the eigenvectors, i.e. to write $x_0 = c_1 x_1 + c_2 x_2$. Letting S be the matrix whose columns are the eigenvectors, we can use matrix notation to write this as $x_0 = Sc$ to obtain $c = S^{-1} x_0$, which has the unique solution $c_1 = 6.25$ and $c_2 = -0.25$.

Aside: Where does A come from?

We are taking A as given, but here is a story about where the entries might come from. Assume that the birth rate in Missouri is 6%, the death rate 1%, and the rate at which people move to Illinois is 2%, which is where we get $A_{11} = 1 + 6\% - 1\% - 2\% = 1.03$ and $A_{21} = 2\%$. Furthermore, we can assume the birth rate in Illinois is 5%, the death rate 1.5%, and the rate at which people move to Missouri is 0.5%, which is where we get $A_{12} = 0.5\%$ and $A_{22} = 1 + 5\% - 1.5\% - 0.5\% = 1.03$.

Population Growth: Solution

So the general solution of $x_t = Ax_{t-1}$ (where A is given above) subject to the initial condition $x_0 = (6, 13)^T$ is

$$
\begin{aligned}
x_t &= A^t x_0 \\
&= A^t(6.25x_1 - 0.25x_2) \\
&= 6.25A^t x_1 - 0.25A^t x_2 \\
&= 6.25 \times 1.04^t x_1 - 0.25 \times 1.02^t x_2 \\
&= 6.25 \times 1.04^t \times \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 0.25 \times 1.02^t \times \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 6.25 \times 1.04^t - 0.25 \times 1.02^t \\ 12.5 \times 1.04^t + 0.5 \times 1.02^t \end{pmatrix}
\end{aligned}
$$

Note that in the long run, the terms with 1.04^t dominate the terms with 1.02^t. Therefore, in the long run both populations grow at a 4% rate, with Illinois' population being about twice Missouri's.

Population Growth: Solution

We can see that the eigenvalue formulation makes the solution much clearer and easier to interpret than simply writing $x_t = A^t x_0$. The eigenvalue decomposition is also an efficient way to compute the value at distant times. In an example with a higher dimension (population in all 50 states or many countries), the eigenvalues and eigenvectors would have to be computed numerically, but the interpretation would be similar, since in the long term the population growth would be the largest eigenvalue less one, and the population proportions would tend to the proportions in the corresponding eigenvector.

Properties of Eigenvalues and Eigenvectors

- Eigenvalues are only defined for square matrices.
- Eigenvalues can be imaginary or complex.
- The trace equals the sum of the eigenvalues.
- The determinant equals the product of the eigenvalues.
- Adding kI to A adds k to each eigenvalue, and leaves the corresponding eigenvectors unchanged.

Properties of Eigenvalues and Eigenvectors of Symmetric Matrices

- All eigenvalues are real.
- There is a complete set of N orthogonal eigenvectors.
- Eigenvectors are orthogonal if they have different eigenvalues.
- A is
  - positive definite iff $(\forall i)\lambda_i > 0$.
  - positive semi-definite iff $(\forall i)\lambda_i \geq 0$.
  - negative semi-definite iff $(\forall i)\lambda_i \leq 0$.
  - negative definite iff $(\forall i)\lambda_i < 0$.

Summary

We have seen only one simple application of eigenvalues and eigenvectors. Another application is the determination of the definiteness of a matrix by looking at the signs of the eigenvalues. Eigenvalues and eigenvectors play a role in differential equations in separating an equation system into independent components, in statistics in looking at independent sources of risk (as in principal components), in probability theory in the analysis of Markov Chains. We have necessarily just touched the surface but hopefully you have some idea of how eigenvalues and eigenvectors can be useful, and they should not be mysterious when you see them again.

$$Ax = \lambda x.$$

$$\det(A - \lambda I) = 0$$

$$(A - \lambda I)x = 0$$