<u>Fin 500J Solutions to Homework 1</u>

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<u>Problem 1</u>. For

$$A = \begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & -1 & -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

compute

$$(1)(A + A^T)B \quad (2)\ Determinant\ of\ C$$

and verify your answers using Matlab.

Solution :

$$(1)\ (A + A^T)B = \begin{pmatrix} 6 & 8 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -8 & 10 & 20 \\ 8 & -2 & 22 & 18 \end{pmatrix}$$

$$(2)|C| = 2 \times (-1)^{1+1} \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} + 4 \times (-1)^{1+2} \times \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + 5 \times (-1)^{1+3} \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}$$

$$= 2 \times 3 + 5 \times (-3) = -9.$$

<u>Problem 2</u>. Invert the coefficient matrix to solve the following systems of equations and verify your answers using Matlab:

(1)

$$2x_1 + x_2 = 5, \quad x_1 + x_2 = 3$$

Solution : The linear system can be written as

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix},$$

so

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 3 \end{pmatrix},$$

using Gaussian Elimination to find the inverse of the coefficient matrix,

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{row(ii)-\frac{1}{2}\times row(i), \frac{1}{2}\times row(ii)} \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{row(i)-row(ii), \frac{1}{2}\times row(i)} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right),$$

therefore,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(2)

$$2x_1 + x_2 = 4, \quad 6x_1 + 2x_2 + 6x_3 = 20, \quad -4x_1 - 3x_2 + 9x_3 = 3.$$

Solution : The linear system can be written as

$$\begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \\ 3 \end{pmatrix},$$

so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 20 \\ 3 \end{pmatrix},$$

using Gaussian Elimination to find the inverse of the coefficient matrix,

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 6 & 2 & 6 & 0 & 1 & 0 \\ -4 & -3 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[row(iii)-row(ii), \frac{1}{2}\times row(iii)]{row(ii)-3\times row(i), row(iii)+2\times row(i)} \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}\times row(i)]{row(ii)-6\times row(iii), (-1)\times row(ii)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & \frac{3}{2} & -1 \\ 0 & 1 & 0 & 13 & -3 & 2 \\ 0 & 0 & 1 & \frac{5}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right),$$

therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6 & \frac{3}{2} & -1 \\ 13 & -3 & 2 \\ \frac{5}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

<u>Problem 3</u>. Determine the definiteness of the following symmetric matrices:

$$A_1 = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 4 & -1 \\ 1 & -1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$$

Solution : (1) A_1 is positive definite because the first order leading principal minor is 5, which is positive, the second order leading principal minor is

$$\begin{vmatrix} 5 & 2 \\ 2 & 4 \end{vmatrix} = 16 > 0$$

and the third order leading principal minor is

$$det(A_1) = 5 \times (-1)^{1+1} \times \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} + 2 \times (-1)^{1+2} \times \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + 1 \times (-1)^{1+3} \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}$$

$$= 5 \times 7 - 2 \times 5 - 6 = 19 > 0.$$

(2) A_2 is indefinite because the first order leading principal minor is 1, which is positive, the second order leading principal minor is

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

but the third order leading principal minor is

$$det(A_2) = 1 \times (-1)^{1+1} \times \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} + 2 \times (-1)^{1+2} \times \begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix}$$

$$= 24 - 25 - 2 \times 12 = -25 < 0.$$

Problem 4. Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix},$$

show

$$(1)\frac{\partial x^T C}{\partial x} = C, \qquad (2)\frac{\partial x^T x}{\partial x} = 2x.$$

Proof : (1) Firstly compute

$$\begin{pmatrix} x_1 & \ldots\ldots & x_n \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^n x_t c_{t1} & \sum_{t=1}^n x_t c_{t2} & \cdots & \sum_{t=1}^n x_t c_{tn} \end{pmatrix},$$

by definition of the partial derivative matrix for vector functions, we have

$$\frac{\partial x^T C}{\partial x} = \begin{pmatrix} \frac{\partial \sum_{t=1}^n x_t c_{t1}}{\partial x_1} & \frac{\partial \sum_{t=1}^n x_t c_{t2}}{\partial x_1} & \cdots & \frac{\partial \sum_{t=1}^n x_t c_{tn}}{\partial x_1} \\ \frac{\partial \sum_{t=1}^n x_t c_{t1}}{\partial x_2} & \frac{\partial \sum_{t=1}^n x_t c_{t2}}{\partial x_2} & \cdots & \frac{\partial \sum_{t=1}^n x_t c_{tn}}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \sum_{t=1}^n x_t c_{t1}}{\partial x_n} & \frac{\partial \sum_{t=1}^n x_t c_{t2}}{\partial x_n} & \cdots & \frac{\partial \sum_{t=1}^n x_t c_{tn}}{\partial x_n} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = C.$$

(2) It is easy to see that

$$x^T x = x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = \sum_{i=1}^n x_i^2.$$

Therefore, by the definition, we have

$$\frac{\partial x^T x}{\partial x} = \begin{pmatrix} \frac{\partial \sum_{i=1}^n x_i^2}{\partial x_1} \\ \frac{\partial \sum_{i=1}^n x_i^2}{\partial x_2} \\ \vdots \\ \frac{\partial \sum_{i=1}^n x_i^2}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{pmatrix} = 2x.$$