OPTIONS and FUTURES
Lecture 4: The Black-Scholes model

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- Black-Scholes option pricing model
- Lognormal price process
- Call price
- Put price
- Using Black-Scholes

Continuous-Time Option Pricing

We have been using the binomial option pricing model of Cox, Ross, and Rubinstein [1979]. In this lecture, we go back to the original modern option pricing model of Black and Scholes [1973]. The mathematical underpinnings of the Black-Scholes model would take a couple of semesters to develop in any formal way, but we can discuss the intuition by viewing it as the limit of the binomial model as the time between trades becomes small.

Black, Fischer, and Myron Scholes (1973) "The Pricing of Options and Corporate Liabilities" *Journal of Political Economy* **81**, 637–654

Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979) "Option Pricing: A Simplified Approach" *Journal of Financial Economics* **7**, 229–263

Lognormal stock price process

The stock price process in the Black-Scholes model is lognormal, that is, given the price at any time, the logarithm of the price at a later time is normally distributed. (Recall that the normal distribution is the familiar "bell curve" from statistics.) It is also known how to do option pricing for a continuous-time model with normally distributed prices, but the lognormal model is more reasonable because stocks have limited liability and cannot go negative. In the basic Black-Scholes model there are no dividends. Over a short period of time, the mean rate of return is μ per unit time and the variance is σ^2 per unit time. For $s < t$, the expected return over the time interval $[s, t]$ is approximately $\mu(t-s)$:

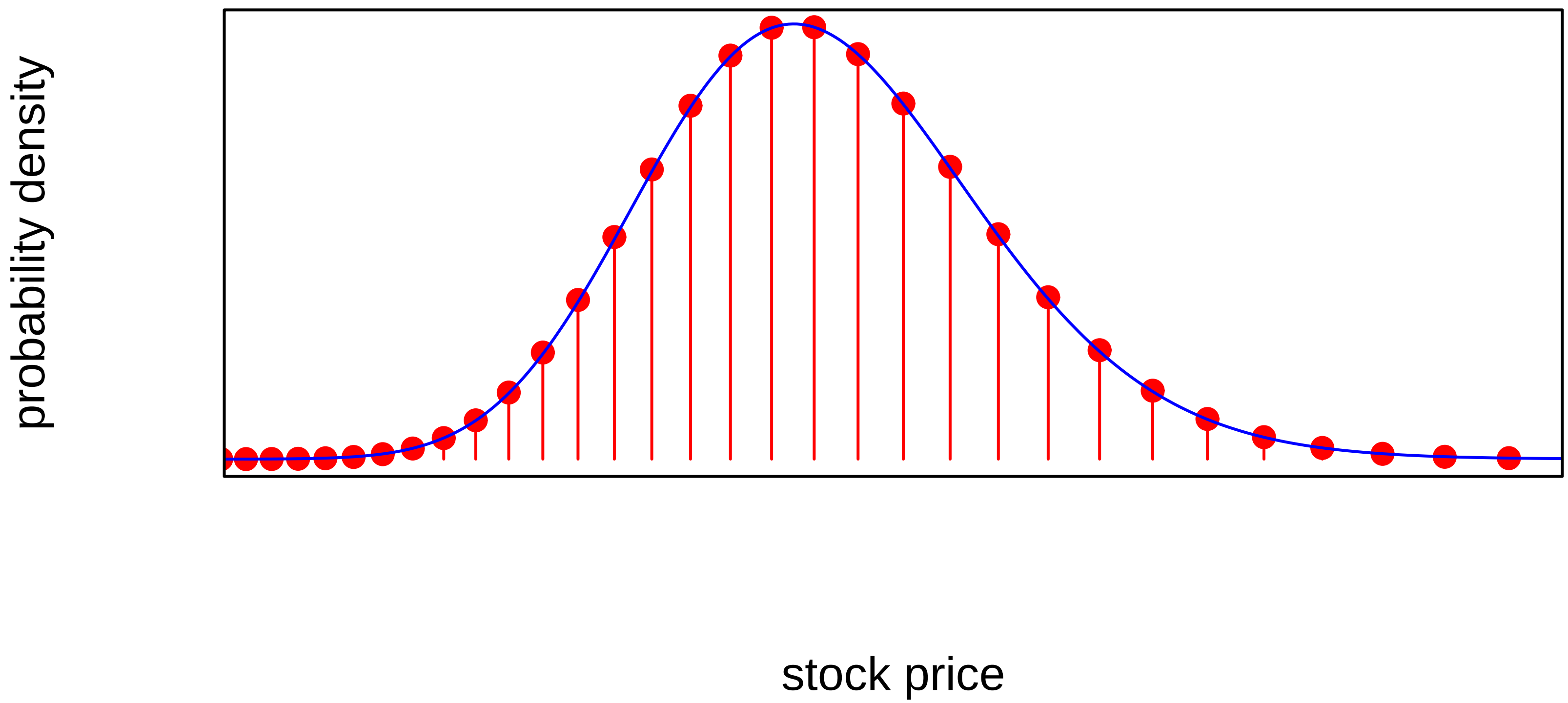
$$E\left[\frac{S_t - S_s}{S_s}\right] \approx \mu(t-s)$$

and the variance is approximately $\sigma^2(t-s)$:

$$E\left[\left(\frac{S_t - S_s}{S_s} - \mu(t-s)\right)^2\right] \approx \sigma^2(t-s).$$

(Note: $\mu(t-s)$ is not exactly the correct mean. However, this does not matter since the squared mean is much smaller than the variance over small time intervals. In fact, it would not matter if we didn't subtract the mean at all.)

Binomial approximates lognormal when time increment is small



Black-Scholes call option pricing formula

The Black-Scholes call price is

$$C(S,T) = SN(x_1) - BN(x_2),$$

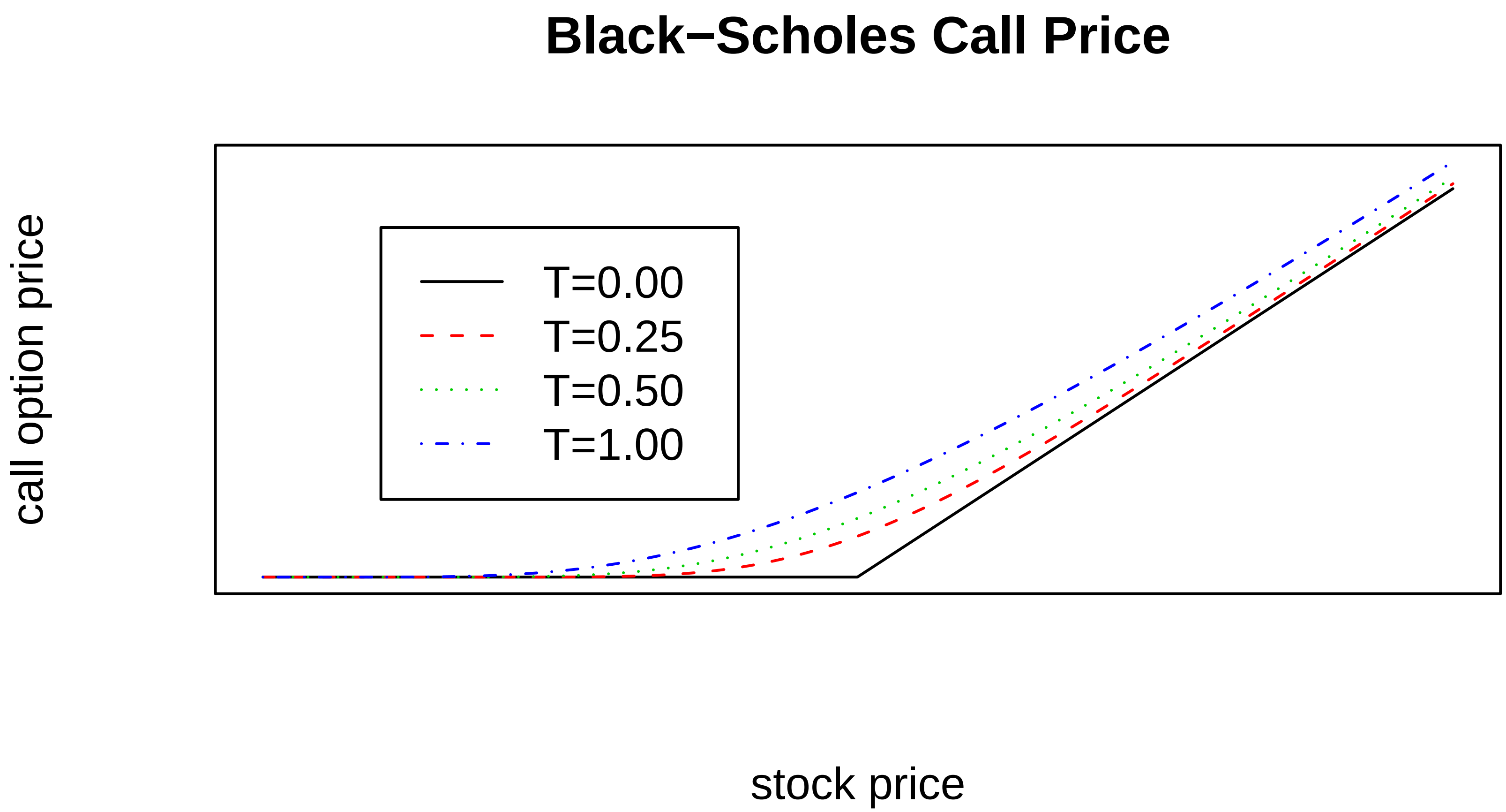
where $N(\cdot)$ is the cumulative normal distribution function, T is time-to-maturity, B is the bond price $Xe^{-r_f T}$,

$$x_1 = \frac{\log(S/B)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T},$$

and

$$x_2 = \frac{\log(S/B)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}.$$

Note that the Black-Scholes option price does not depend on the mean return of the stock. This is because the change to risk-neutral probabilities changes the mean but not the variance. Note that these prices are for European options on stocks paying no dividends.


Black−Scholes Call Price
T=0.00
T=0.25
T=0.50
T=1.00
call option price
stock price

Intuition: gambler's rule*

When a call option ends in the money at maturity T periods from now, its value is $S_T - X_T$, which has value $S - Xe^{-r_f T} = S - B$ in the notation above. If the probability of exercise were π and chosen independent of everything in the economy, the option value would be $S\pi - B\pi$. However, we can skew the odds in our favor if we pick and choose when to exercise, which skews the probabilities: $SN(x_1) - BN(x_2)$ where $x_1 > x_2$. As maturity approaches, both probabilities tend to 1 if the option is in the money or 0 if the option is out of the money. In particular, if we look at x_1 and x_2, they both go to $+\infty$ as the moneyness S/B of the option increases or goes down to $-\infty$ as the moneyness decreases. The difference between x_1 and x_2 is the greatest when volatility and time-to-maturity are greatest.

*fanciful description due to Stephen Brown

Where does Black-Scholes come from?

The Black-Scholes formula can be derived as the limit of the binomial pricing formula as the time between trades shrinks, or directly in the continuous time model using an arbitrage argument.

The option value is a function of the stock price and time, and the local movement in the stock price can be computed using a result called Itô's lemma, which is an extension of the chain rule from calculus. The standard version of the chain rule does not work, because the stock price in the lognormal model is not differentiable (and cannot be or else stock price would be locally predictable implying the existence of an arbitrage). Even if a function of the stock has zero derivative at a point, its expected rate of increase can be positive due to the volatility of local price movements.

Once Itô's lemma is used to calculate the local change in the option value in term of derivatives of the function of stock price and time, absence of arbitrage implies a restriction on the derivatives of the function (in economic terms, risk premium is proportional to risk exposure), essentially similar to the per-period hedge in the binomial model. The absence of arbitrage implies a differential equation that is solved subject to the boundary condition of the known option value at the end.

What are the two terms?

I am not sure it widely known, but the two terms in Black-Scholes call formula are prices of digital options. The first term $SN(x_1)$ is the price of a digital option that pays one share of stock at maturity when the stock price exceeds X: this is a digital option if we measure payoffs in terms of the stock price (this is called using the stock as numeraire and is like a currency conversion). The second term $-XN(x_2)$ is the price of a short position in a digital option that pays X at maturity when the stock price exceeds X.

There is a slightly mystical result that the two terms also represent the portfolio we hold to replicate the option if we want to perform an arb. We can create the call option payoff at the end by holding $SN(x_1)$ long in stocks and $BN(x_2)$ short in bonds (with trading to vary this continuously as time passes and the stock price evolves).

In-class exercise: Black-Scholes put price

Derive the Black-Scholes put price.

hint: Use the known form of the Black-Scholes call price ($SN(x_1) - BN(x_2)$) and put-call parity ($C + B = P + S$).


Black−Scholes Put Price
put option price
stock price
T=0.00
T=0.25
T=0.50
T=1.00

Using the Black-Scholes formula

The Black-Scholes model is usually the model of choice when working with a plain vanilla European option pricing application. The binomial model is more flexible and is a better choice for inclusion of a nontrivial American feature, realistic dividends, and other complications.

The simplest way to obtain the Black-Scholes call price is to use available tables, a spreadsheet, or a financial calculator. Or, compute the value directly using the formula. The value of $N(\cdot)$ is available in tables or from the approximation in the next slide. Be sure to use the *natural logarithm* function to compute $log(S/B)$.

Cumulative Normal Distribution Function

If you want to calculate the formula yourself, there are lots of good approximations to the cumulative normal distribution function. For example, you can use the following procedure for $x \leq 0$. For $x \geq 0$, use the procedure on $-x$ and the formula $N(x) = 1 - N(-x)$. First compute t, defined by $t \equiv 1/(1 - 0.33267x)$. Then use t and x to compute $N(x) \approx (0.4361836t - 0.1201676t^2 + 0.937298t^3)\exp(-x^2/2)/\sqrt{2\Pi}$, where $\Pi \approx 3.1415926535$ is the familiar numerical constant.

If x very close to zero (say $-.25 < x < .25$), as it will be for near-the-money options at short maturities, $N(x) \approx .5 + x/\sqrt{2\Pi} \approx .5 + 0.39894x$. Consequently, near-the-money call options are worth about $\frac{S-B}{2} + .4\frac{S+B}{2}\sigma\sqrt{T}$. This is a handy approximation for a meeting or job interview where a quick approximate answer is useful.

What interest rate to use

The original Black-Scholes derivation assumes that the interest rate is always constant and is the same for all maturities. Of course, the riskless interest rate is not constant, and bonds of different maturities have different yields.

In the Black-Scholes formula, the interest rate always appears in $e^{-r_f \tau}$, which is the price of a riskless pure discount bond with a face value of 1 and a maturity τ periods from now. The generalizations of Black-Scholes suggest that we should use the price of a discount bond maturing with the option and a face equal to the strike price. Traditionally, prices of Treasury securities were used, but now most practitioners use LIBOR instead as Treasuries are viewed as less liquid.

What variance to use

The choice of variance estimate is even more important than the choice of interest rate in most option pricing applications. Getting the variance wrong can have a big impact on the computed price, and an equally big impact on the effectiveness of a hedge. The two most common methods of determining the appropriate variance are historical estimation and implied variance.

In the historical estimation approach, simply look at the variance of historical returns and then adjust for the length of the time horizon. For technical reasons, it is best to use "log" returns (that is, to compute the sample variance of $log(1+return)$). Be sure that your returns are computed correctly, and that you have adjusted properly for any dividends or stock splits. Weekly returns seem like an ideal choice: longer returns give less accurate estimates and are more subject to changes in variance over time, while shorter returns are more contaminated by the bid-ask spread, liquidity distortions, and non-trading effects. After the variance is computed, do the proper adjustment to convert to annual terms. For example, a weekly variance of returns of $.0032$ corresponds to an annual variance of $52 \times 0.0032 \approx 0.16$ or an annual standard deviation of $\sqrt{0.16} = 40\%$.

What Variance to Use (continued)

The implied variance approach to computing variance is a way of reading market participants' assessment of variance. It requires there to be quoted prices available for options on the stock on which the option is written, or for options on a related instrument. The idea is to look at what variance (the implied variance) would be consistent with the option prices you see in the market. In doing so, it is important to get option and stock prices that are as nearly contemporaneous as possible, which is particularly difficult if the stock or its options are thinly traded. This approach is used mostly in active markets (for example, for stock index futures). Practitioners often use implied variances as an indicator of the state of these markets.

Of course, individual judgment also enters the equation. For example, if you believe that historical variances were estimated on a particularly turbulent period and that we are in a quieter period, then you should adjust the estimate downward. Or, if you obtain an implied variance that doesn't seem sensible, you should obviously consider whether the price quotes (and your calculations) are reliable.

Warning: Black-Scholes is for underlying *assets*

Do not just plug in an interest rate, vol, or other number where the stock price goes in Black-Scholes! If you make this common mistake, the result is nonsense and any decisions based on this mistake are likely to cost you money. The Black-Scholes model assumes in its derivation that we can set up a hedge by going long or short the stock to enforce the pricing. However, there is no asset we can buy today whose price today is the interest rate today and whose price tomorrow is the interest rate tomorrow!

Wrap-up

- Black-Scholes uses the same economics as the binomial model
- continuous time with a lognormal price process
- closed form solutions for European call and put prices
- for the interest rate, use LIBOR
- for the vol, used historical, implied vol, and/or your own judgment